EFFECT OF THE VERTICAL MAGNETIC FIELD ON RAYLEIGH-TAYLOR INSTABILITY FOR INCOMPRESSIBLE FLUIDS WITH AN UPPER FREE SURFACE

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ABSTRACT. In this paper, we consider a nonhomogeneous incompressible magnetohydrodynamic fluid in a horizontally periodic domain, being bounded above by a free moving boundary and bounded below by a fixed bottom. The governing equations are the gravity-driven incompressible Navier-Stokes equations interacting with a magnetic field and after using the Lagrangian transformation, we write the main equations in a perturbed form in a fixed domain. The goal of this paper is to study the influence of the vertical magnetic field on the nonlinear Rayleigh-Taylor (RT) instability result of a smooth increasing RT density profile. Precisely, we prove that the nonlinear problem departing from the hydrostatic equilibrium is nonlinearly unstable under L^2-norm as the strength $|\mathrm{m}|$ of the steady vertical magnetic field is lower than the critical value m_c, improving the nonlinear RT instability result of F. Jiang and S. Jiang [12] under H^2-norm. Our nonlinear result refines the abstract framework of Guo and Strauss [5] and also of Grenier [8] with a wide class of initial data for the nonlinear problem, based on the finding of infinitely many normal modes to the linearized equations via the operator method initiated by Lafitte and Nguyen [22].

CONTENTS

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1. INTRODUCTION

The Rayleigh–Taylor (RT) instability, studied first by Lord Rayleigh in [28] and then Taylor [29] is well known as a gravity-driven instability in two semi-infinite inviscid and incompressible fluids when the heavy one is on top of the light one. It has attracted much attention due to both its physical and mathematical importance. Two applications worth mentioning are implosion of inertial confinement fusion capsules [23] and core-collapse of supernovae [27]. For a detailed physical comprehension of the linear RT instability, we refer to three survey papers [19, 33, 34]. Mathematically speaking, the nonlinear study of classical RT instability is proven by Desjardins and Grenier [3]. For the inviscid and incompressible fluid with a smooth density profile, the classical RT instability was investigated by Lafitte [21], by Guo and Hwang [4] and by Helffer and Lafitte [15]. For the viscous linear RT instability, one of the first studies can be seen in the book of Chandrasekhar [2, Chapter X]. He considers two uniform viscous fluid separated by a horizontal boundary and generalize the classical result of Rayleigh and Taylor. We refer the readers to mathematical viscous linear/nonlinear RT studies for two (in-)compressible channel flows in [6], [32] and [10]. For the incompressible fluid with a smooth density profile, we mention the results of Jiang et al. [13], of Lafitte and Nguyen [22], and of Nguyen [26] respectively.

In this paper, we study the magnetohydrodynamic (MHD) influence on the RT instability of an increasing RT density profile. Owing to the presence of the magnetic field, numerous results the RT instability of continuous incompressible fluids cannot be extended straightforwardly. Let us mention some previous results on the effect of magnetic field to linear Rayleigh-Taylor instability. In 1954, Kruskal and Schwarzschild [20] investigated the effect of the horizontal magnetic field $M = \mathrm{m}e_1$ ($e_1 = (1, 0, 0)^T$) to the linear instability problem for stratified MHD fluids on a horizontally periodic domain. After that, the linear RT instability influenced by a vertical magnetic field $M = \mathrm{m}e_3$ ($e_3 = (0, 0, 1)^T$) was proven for a continuous incompressible MHD fluid by Hide [16] (see also the book of Chandrasekhar [2, Chapter 4]). Considering the linearized problem in a bounded domain, Jiang and Jiang [11] obtained the threshold m_c of the vertical magnetic field for the linear instability as $|\mathrm{m}| < \mathrm{m}_c$ and stability as $|\mathrm{m}| > \mathrm{m}_c$.

One of the first nonlinear study on the continuous incompressible magnetic RT instability, was given by Hwang [17] for the inviscid case. Then, Jiang, Jiang and Wang [14] extended Hwang's result to the viscous case by following the framework of Guo and Strauss [5]. Such magnetic RT instability was also investigated by Wang for stratified incompressible MHD fluids [31]. For MHD fluids in a horizontally periodic domain with finite height, Jiang and Jiang [12] studied the stabilizing effect of the vertical magnetic field in the supercritical regime $|\mathrm{m}| > \mathrm{m}_c$ and for MHD fluids in a bounded domain, they proved the nonlinear instability under H^2-norm of the vertical magnetic field in the subcritical regime $|\mathrm{m}| < \mathrm{m}_c$, extending [11].

This motivates us to show in this paper an improved result of the nonlinear RT instability in the subcritical regime $|\mathrm{m}| < \mathrm{m}_c$ under L^2-norm. Let us present the precise formulation as follows.

We consider the fluid lying on a vertical gravity field, below some smooth interface separating it from air. That means, at time t, the fluid occupies a horizontally periodic domain

$$\Omega(t) = \{x = (x_1, x_2, x_3) : (x_1, x_2) \in (2\pi L\mathbb{T})^2, -h < x_3 < d(t, x_1, x_2)\}, \tag{1.1}$$

where d is smooth enough function of its arguments. We denote the upper free surface

$$\Gamma(t) = \{x_3 = d(t, x_1, x_2), (x_1, x_2) \in (2\pi L\mathbb{T})^2\}$$

and the fixed bottom $\Gamma_h = (2\pi L\mathbb{T})^2 \times \{x_3 = -h\}$. The fluid dynamic is described by the gravity-driven incompressible Navier-Stokes equations interacting with the magnetic field $\tilde{M}$, that read as

$$\begin{cases} \partial_t\tilde{\rho} + \mathrm{div}(\tilde{\rho}\tilde{u}) = 0 & \text{in } \Omega(t), \\ \partial_t(\tilde{\rho}\tilde{u}) + \mathrm{div}(\tilde{\rho}\tilde{u} \otimes \tilde{u}) + \mathrm{div}\tilde{\mathbb{S}} = -g\tilde{\rho}e_3 & \text{in } \Omega(t), \\ \partial_t\tilde{M} = \nabla \times (\tilde{u} \times \tilde{M}) & \text{in } \Omega(t), \\ \mathrm{div}\tilde{u} = 0, \quad \mathrm{div}\tilde{M} = 0 & \text{in } \Omega(t). \end{cases}$$

The unknowns $\tilde{\rho} := \tilde{\rho}(t, y)$, $\tilde{u} := \tilde{u}(t, y)$ and $\tilde{p} := \tilde{p}(t, y)$ denote the density, the velocity and the pressure of the fluid, respectively. In the second equation, $-g\tilde{\rho}e_3$ is the gravity field with $g > 0$ the acceleration of gravity and e_3 the vertical unit vector. The stress tensor $\tilde{\mathbb{S}}$ consist of both fluid and magnetic parts is given by

$$\tilde{\mathbb{S}} = -\mu(\nabla\tilde{u} + (\nabla\tilde{u})^T) + \tilde{p}\mathrm{Id} + \frac{|\tilde{M}|^2}{2}\mathrm{Id} - \tilde{M} \otimes \tilde{M}.$$

On the free surface $\Gamma(t)$, we have the dynamic boundary condition without any effect of surface tension (p_{atm} is the given atmospheric pressure)

$$\tilde{\mathbb{S}}\tilde{n} = p_{atm}\tilde{n},$$

and the kinematic boundary condition

$$\partial_t d = \tilde{u}_3 - \tilde{u}_1\partial_1 d - \tilde{u}_2\partial_2 d.$$

At the fixed bottom Γ_h, we enforce the condition that the fluid velocity vanishes, $\tilde{u} = 0$. We formulate the governing equations

$$\begin{cases} \partial_t\tilde{\rho} + \tilde{u} \cdot \nabla\tilde{\rho} = 0 & \text{in } \Omega(t), \\ \tilde{\rho}(\partial_t\tilde{u} + \tilde{u} \cdot \nabla\tilde{u}) + \mathrm{div}\tilde{\mathbb{S}} = -g\tilde{\rho}e_3 & \text{in } \Omega(t), \\ \partial_t\tilde{M} + \tilde{u} \cdot \nabla\tilde{M} = \tilde{M} \cdot \nabla\tilde{u} & \text{in } \Omega(t), \\ \mathrm{div}\tilde{u} = 0, \quad \mathrm{div}\tilde{M} = 0 & \text{in } \Omega(t), \\ \tilde{\mathbb{S}}\tilde{n} = p_{atm}\tilde{n} & \text{on } \Gamma(t), \\ \partial_t d = \tilde{u}_3 - \tilde{u}_1\partial_1 d - \tilde{u}_2\partial_2 d & \text{on } \Gamma(t), \\ \tilde{u} = 0 & \text{on } \Gamma_h. \end{cases} \tag{1.2}$$

To complete the statement of the problem (1.2), we must specify the initial conditions. We suppose that the initial surface $\Gamma(0)$ is given, i.e. $d|_{t=0} = d_0$ is given on $(2\pi L\mathbb{T})^2$, which yields the open set $\Omega(0)$. Hence, on $\Omega(0)$, we specify the initial data for the density $\tilde{\rho}(0) : \Omega(0) \to \mathbf{R}$, the velocity $\tilde{u}(0) : \Omega(0) \to \mathbf{R}^3$ and the magnetic field $\tilde{M}(0) : \Omega(0) \to \mathbf{R}^3$.

We now construct an equilibrium state to the system (1.2). Let $d \equiv 0$, we define the equilibrium surface

$$\Gamma_0 = (2\pi L\mathbb{T})^2 \times \{0\},$$

yielding $d \equiv 0$ and $\tilde{n} \equiv e_3$, and thus define the fixed domain

$$\Omega = (2\pi L\mathbb{T})^2 \times (-h, 0).$$

Hence, let m is an arbitrary constant, we have that a density profile $\rho_0(x_3)$, a zero velocity $u \equiv 0$ and a vertical magnetic field $\bar{M} = \mathrm{m}e_3$ (m is a constant) define a hydrostatic equilibrium with pressure gradient balancing out the gravity field,

$$p_0(0) = p_{atm} \quad \text{and} \quad \nabla p_0 = -g\rho_0 e_3, \text{ i.e. } p_0' = -g\rho_0 (' = \frac{d}{dx_3}). \tag{1.3}$$

We assume that

$$\rho_0' > 0 \tag{1.4}$$

and denote by

$$0 < \rho_- = \rho_0(-h) < \rho_0(0) = \rho_+ < +\infty. \tag{1.5}$$

In that way, we have heavier fluid above lighter one, and we are thus in the situation of Rayleigh-Taylor instability occurs. The goal of this paper is to show that under the assumptions (1.4), (1.5) and under the effect of magnetic field, the nonlinear RT instability in MHD flows happens in the subcritical regime of vertical magnetic number

$$|\mathrm{m}| < \mathrm{m}_c := \sqrt{\max_{\phi \in H^1((-h,0))} \frac{g\int_{-h}^0 \rho_0' \phi^2 - g\rho_+ \phi^2(0)}{\int_{-h}^0 (\phi')^2}}. \tag{1.6}$$

That means, we construct initial data of small size δ, giving rise to a solution defined up to some time T^δ, and which at that time has L^2-norm bounded from below by a fixed constant (independent of δ). We refer to Section 2 for the precise statement of our main theorems and describe here our strategy of the proof.

The first step in our proof is to construct a solution of the linearization at this stationary solution of the nonlinear equations. We want this solution of the linearized equations (2.14) to have growing normal modes and we look for it schematically as $U(t, x) = e^{\lambda t} V(x)$, where λ is positive. The profile V is taken as an oscillatory function of the horizontal variables (x_1, x_2), with mode $\mathbf{k} = (k_1, k_2)$, the x_3-dependence being given in terms of unknown functions of $(\mathbf{k}, x_3)$. Then U is a solution of the linearized equations if some function $x_3 \to \phi(\mathbf{k}, x_3)$ (from which U maybe reconstructed) solves a fourth order ODE on the interval $(-h, 0)$, depending on $\mathbf{k}$ and λ. Our first theorem asserts that, as $|\mathrm{m}| < \mathrm{m}_c$, one may find *infinitely many* solutions to that ODE and thus get *infinitely many* normal mode solutions of the linearized equations. The line of investigation is the same as in [22], where the case of a viscous nonhomogeneous incompressible fluid in the whole space has been treated.

The second part of the paper is to devoted to the proof of nonlinear instability. The spectral analysis allows us to study the fully nonlinear perturbation equations (2.13). To this purpose, we follow the same procedure as in [24] for RT problem in an infinite strip with Navier-slip boundary conditions,

Step 1. establish some *a priori* energy estimates to the nonlinear equations,
Step 2. formulate a linear combination of normal modes to the linearized equations (2.14) to set its value at initial time $t = 0$ of size $0 < \delta \ll 1$ as an initial datum to the nonlinear perturbation equations,
Step 3. obtain the difference between the local exact solution and the approximate solution in Step 2 and exploit some energy estimates for the difference,
Step 4. deduce the bound in time of the difference functions and prove the nonlinear instability.

Our nonlinear study is inspired by the abstract frameworks of Guo and Strauss [5] and of Grenier [8]. In the above frameworks, only the maximal normal mode was used in Step 2 to approximate the nonlinear equations. Let us emphasize that, our nonlinear results show that a *wide class* of initial data (related to a linear combination of normal modes) to the nonlinear problem departing from the equilibrium is formulated in Step 2 and it gives rise to the nonlinear instability.

To finish the introduction part, we introduce the organization of this paper. In Section 2, from the formulation in Eulerian coordinates of the governing equations (1.2), we derive the formulation in Lagrangian coordinates, see (2.13). We introduce our main results, Theorem 2.1 describing the spectral analysis of the linearized equations (2.14) and Theorem 2.2 proving the linear instability in the subcritical regime $|\mathrm{m}| < \mathrm{m}_c$. The proof of Theorems 2.1, 2.2 will be shown in Section 3. In Section 4, we construct the a priori energy estimates to the nonlinear equations and in the last part, Section 5, we conclude the nonlinear instability, Theorem 2.3, still in the subcritical regime $|\mathrm{m}| < \mathrm{m}_c$.

2. Reformulation in Lagrangian coordinates and main results

2.1. **Reformulation in Lagrangian coordinates.** The movement of the free boundary $\Gamma(t)$ and the domain $\Omega(t)$ raises numerous mathematical difficulties. To handle that, we will switch to coordinates in which the domain stay fixed in time. Since we are interested in the nonlinear instability of the equilibrium state, we will use Ω as the equilibrium domain. We assume that there exists an invertible mapping $\zeta_0 : \Omega \to \Omega(0)$ such that

$$\Gamma_0 = \zeta_0(\Gamma) \quad \text{and} \quad \det(\nabla\zeta_0) = 1.$$

Define the flow maps ζ as the solution to

$$\begin{cases} \partial_t\zeta(t,x) = u(t,\zeta(t,x)), \\ \zeta(0,x) = \zeta_0(x). \end{cases} \tag{2.1}$$

We think of the Eulerian coordinates as $(t,y) \in \mathbf{R}_+ \times \Omega(t)$ with $y = \zeta(t,x)$, whereas we think of Lagrangian coordinates as the fixed $(t,x) \in \mathbf{R}_+ \times \Omega$. In order to switch back and forth from Lagrangian to Eulerian coordinates we assume that $\zeta(t,\cdot)$ are invertible and

$$\Omega(t) = \zeta(t,\Omega), \quad \Gamma(t) = \zeta(t,\Gamma_0), \quad \Gamma_h = \zeta(t,\Gamma_h).$$

If $\zeta - \mathrm{Id}$ is sufficiently small in an appropriate Sobolev norm, then ζ is a diffeomorphism, i.e.

$$J = \det(\nabla\zeta) \neq 0.$$

This allows us to switch back and forth from Lagrangian to Eulerian coordinates and transform the problem (1.2) to one in the fixed spatial domain Ω. To this purpose, we define the matrix $\mathcal{A} = (\mathcal{A}_{ij})_{1\leqslant i,j\leqslant 3}$ via its transpose $\mathcal{A}^T := (\nabla\zeta)^{-1}$ and define the following differential operators

$$\nabla_{\mathcal{A}} f := (\mathcal{A}_{1k}\partial_k f, \mathcal{A}_{2k}\partial_k f, \mathcal{A}_{3k}\partial_k f)^T, \quad \mathrm{div}_{\mathcal{A}}(X_1,X_2,X_3)^T := \mathcal{A}_{lk}\partial_k X_l, \quad \Delta_{\mathcal{A}} f := \mathrm{div}_{\mathcal{A}}\nabla_{\mathcal{A}} f,$$

where we have used the Einstein convention of summation over repeated indices. We write

$$\mathcal{N} = \partial_1\zeta \times \partial_2\zeta|_{\Gamma_0} = J\mathcal{A}e_3|_{\Gamma_0} \tag{2.2}$$

for the non-unit normal to $\Gamma(t)$ and

$$\mathbb{S}_{\mathcal{A}}(p,u) := -\mu D_{\mathcal{A}}u + p\mathrm{Id}, \quad (D_{\mathcal{A}}u)_{ij} = \mathcal{A}_{ik}\partial_k u_j + \mathcal{A}_{jk}\partial_k u_i.$$

With the above notations, we define the Lagrangian unknowns on $(\tilde{\rho},\tilde{u},\tilde{p},\tilde{M})$ by the compositions

$$(\rho,u,p,M)(t,x) = (\tilde{\rho},\tilde{u},\tilde{p} + \frac{|\tilde{M}|^2}{2},\tilde{M})(t,\zeta(t,x))$$

and derive from (1.2), the evolution equations for (ρ, u, p, M) in Lagrangian coordinates as follows

$$\begin{cases} \partial_t \zeta = u, \quad \partial_t \rho = 0 & \text{in } \Omega, \\ \rho \partial_t u - \mu \Delta_{\mathcal{A}} u + \nabla_{\mathcal{A}} p = M \cdot \nabla_{\mathcal{A}} M - g\rho e_3 & \text{in } \Omega, \\ \partial_t M - M \cdot \nabla_{\mathcal{A}} u = 0 & \text{in } \Omega, \\ \mathrm{div}_{\mathcal{A}} u = \mathrm{div}_{\mathcal{A}} M = 0 & \text{in } \Omega, \\ \mathbb{S}_{\mathcal{A}}(p, u)\mathcal{N} = (M \cdot \mathcal{N}) M & \text{on } \Gamma_0, \\ u = 0 & \text{on } \Gamma_h. \end{cases} \tag{2.3}$$

Clearly, $(2.3)_2$ implies that $\rho(t, \cdot)$ is constant along time. We expect that converges to the equilibrium density profile $\bar\rho(x_3)$ as $t \to \infty$. Hence,

$$\rho(t, x) = \rho_0(x_3) \quad \text{for any } (t, x). \tag{2.4}$$

Next, we eliminate M by expressing it in terms of ζ, and this can be achieved in the same manner as in [12, 31]. From a direct computation, we have that

$$\partial_t J = J \mathrm{div}_{\mathcal{A}} u = 0. \tag{2.5}$$

That implies $J \equiv 1$ in Ω. Next, applying $\mathcal{A}^T$ to $(2.3)_4$, we obtain

$$\mathcal{A}_{ji}\partial_t M_j = \mathcal{A}_{ji} M_k \mathcal{A}_{kl} \partial_l u_j = \mathcal{A}_{ji} M_k \mathcal{A}_{kl} \partial_t(\partial_l \zeta_j) = -\partial_t \mathcal{A}_{ji} M_k \mathcal{A}_{kl} \partial_l \eta_j = -M_j \partial_t \mathcal{A}_{ji}.$$

This yields

$$\partial_t(\mathcal{A}^T M) = 0, \tag{2.6}$$

hence $\mathcal{A}_{jl} M_j = \mathcal{A}^0_{jl} M^0_j$. We get further

$$M_i(t) = \partial_l \zeta_i(t) \mathcal{A}_{jl}(0) M_j(0), \quad \text{i.e.} \quad M(t) = \nabla\zeta(t) \mathcal{A}^T(0) M(0). \tag{2.7}$$

To obtain the asymptotic stability of the magnetic RT equilibrium state in time, we naturally expect that

$$(\zeta, M) \text{ converges to } (x, \bar M) \text{ as } t \to \infty.$$

Thus, we formally obtain from (2.7) that $\mathcal{A}^T(0) M(0) = \bar M$, yielding

$$M = \bar M \cdot \nabla \zeta \tag{2.8}$$

and

$$M \cdot \nabla_{\mathcal{A}} M = M_j \mathcal{A}_{jk} \partial_k M = \bar M_k \partial_k (\bar M_l \partial_l \zeta) = (\bar M \cdot \nabla)^2 \zeta. \tag{2.9}$$

Note that, it follows from (2.2), (2.5) and (2.6) that

$$\partial_t \mathrm{div}_{\mathcal{A}} M = J^{-1} \partial_t \mathrm{div}(J \mathcal{A}^T M) = 0$$

and

$$\partial_t(M \cdot \mathcal{N}) = \partial_t(M \cdot J\mathcal{A} e_3) = \partial_t(J M^T \mathcal{A} e_3) = 0 \quad \text{on } \Gamma_0.$$

Hence,

$$\mathrm{div}_{\mathcal{A}} M = \mathrm{div} \bar M = 0 \text{ in } \Omega \text{ and } M \cdot \mathcal{N} = \bar M \text{ on } \Gamma_0. \tag{2.10}$$

Summing up the above calculations (2.4) (2.8), (2.9) and (2.10), and letting

$$\eta = \zeta - x, \quad \mathcal{A}^T = (Id + \nabla \eta)^{-1},$$

we can transform (2.3) into a Navier–Stokes system in Ω with two force terms induced by the flow map and the equilibrium density profile ρ_0:

$$\begin{cases} \partial_t \eta = u & \text{in } \Omega, \\ \rho_0 \partial_t u - \mu \Delta_{\mathcal{A}} u + \nabla_{\mathcal{A}} p = \mathrm{m}^2 \partial_3^2 \zeta - g\rho_0 e_3 & \text{in } \Omega, \\ \mathrm{div}_{\mathcal{A}} u = 0 & \text{in } \Omega, \\ \mathbb{S}_{\mathcal{A}}(p,u)\mathcal{N} = \mathrm{m}^2 \partial_3 \eta & \text{on } \Gamma_0, \\ u = 0 & \text{on } \Gamma_h. \end{cases} \tag{2.11}$$

Let

$$\tilde{\rho}_0 = \rho_0(x_3 + \eta_3) \quad \text{and} \quad \tilde{p}_0 = p_0(x_3 + \eta_3),$$

the equilibrium state (1.3) in Lagrangian coordinates reads as $\nabla_{\mathcal{A}} \tilde{p}_0 = -g\tilde{\rho}_0 e_3$. That implies

$$\begin{aligned} \nabla_{\mathcal{A}} p_0 &= -g\tilde{\rho}_0 e_3 - \nabla_{\mathcal{A}}(\tilde{p}_0 - p_0) \\ &= -g\tilde{\rho}_0 e_3 - \nabla_{\mathcal{A}}(p_0'(x_3)\eta_3) + \mathcal{Q}_p \\ &= -g\tilde{\rho}_0 e_3 + g\nabla_{\mathcal{A}}(\rho_0 \eta_3) + \mathcal{Q}_p, \end{aligned}$$

where the quadratic term $\mathcal{Q}_p$ is given by

$$\mathcal{Q}_p := -\nabla_{\mathcal{A}}\Big(\eta_3^2 \int_0^1 (1-s)\frac{d^2}{ds^2} p_0(x_3 + s\eta_3)ds\Big).$$

We now define the modified pressure

$$q = p - p_0 + g\rho_0 \eta_3$$

and obtain that

$$\nabla_{\mathcal{A}} p + g\rho_0 e_3 = \nabla_{\mathcal{A}} q + g(\rho_0 - \tilde{\rho}_0)e_3 - \mathcal{Q}_p = \nabla_{\mathcal{A}} q - g\rho_0' \eta_3 e_3 - \mathcal{Q}_p - \mathcal{Q}_g,$$

where the quadratic term $\mathcal{Q}_g$ is given by

$$\mathcal{Q}_g := g\Big(\eta_3^2 \int_0^1 (1-s)\frac{d^2}{ds^2} \rho_0(x_3 + s\eta_3)ds\Big)e_3.$$

We deduce the evolution equations for (η, u, q) as

$$\begin{cases} \partial_t \eta = u & \text{in } \Omega, \\ \rho_0 \partial_t u - \mu \Delta_{\mathcal{A}} u + \nabla_{\mathcal{A}} q - \mathrm{m}^2 \partial_3^2 \eta - g\rho_0' \eta_3 e_3 = \mathcal{Q}_p + \mathcal{Q}_g & \text{in } \Omega, \\ \mathrm{div}_{\mathcal{A}} u = 0 & \text{in } \Omega, \\ \mathbb{S}_{\mathcal{A}}(q,u)\mathcal{N} = \mathrm{m}^2 \partial_3 \eta + g\rho_+ \eta_3 \mathcal{N} & \text{on } \Gamma_0, \\ u = 0 & \text{on } \Gamma_h. \end{cases} \tag{2.12}$$

Hence, around the trivial state $U = (\eta, u, q) \equiv 0$, we consider from now on the following homogeneous linear form

$$\begin{cases} \partial_t \eta = u & \text{in } \Omega, \\ \rho_0 \partial_t u - \mu \Delta u + \nabla q - \mathrm{m}^2 \partial_3^2 \eta - g\rho_0' \eta_3 e_3 = \mathcal{Q}_1 & \text{in } \Omega, \\ \mathrm{div} u = \mathcal{Q}_2 & \text{in } \Omega, \\ \mathbb{S}(q,u)e_3 = \mathrm{m}^2 \partial_3 \eta + g\rho_+ \eta_3 e_3 + \mathcal{Q}_3 & \text{on } \Gamma_0, \\ u = 0 & \text{on } \Gamma_h, \end{cases} \tag{2.13}$$

where the nonlinear terms $\mathcal{Q}_1$, $\mathcal{Q}_2$ and $\mathcal{Q}_3$ are given by

$$\begin{aligned}\mathcal{Q}_1 &:= \mu(\Delta_{\mathcal{A}} u - \Delta u) - (\nabla_{\mathcal{A}} q - \nabla q) + \mathcal{Q}_p + \mathcal{Q}_g,\\ \mathcal{Q}_2 &:= \mathrm{div} u - \mathrm{div}_{\mathcal{A}} u,\\ \mathcal{Q}_3 &:= (q - g\rho_+\eta)\mathrm{Id}\cdot(e_3 - \mathcal{N}) - \mu\mathbb{S}u e_3 + \mu(\mathbb{S}_{\mathcal{A}} u)\mathcal{N}\end{aligned}$$

To investigate the nonlinear RT instability to (1.2) in the subcritical regime of vertical magnetic number, we move to prove the nonlinear instability of the trivial state $U = (\eta, u, q) \equiv 0$ to the nonlinear equations (2.13) in the above regime.

2.2. **The linear instability.**

The linearized equations of (2.13) are

$$\begin{cases}\partial_t \eta = u & \text{in } \Omega,\\ \rho_0\partial_t u - \mu\Delta u + \nabla q - \mathrm{m}^2\partial_3^2\eta = g\rho_0'\eta_3 e_3, & \text{in } \Omega,\\ \mathrm{div} u = 0 & \text{in } \Omega,\\ (q\mathrm{Id} - \mu\mathbb{S}u)e_3 = \mathrm{m}^2\partial_3\eta + g\rho_+\eta_3 e_3 & \text{on } \Gamma_0,\\ u = 0 & \text{on } \Gamma_h.\end{cases} \tag{2.14}$$

As in [2, Chapter XI], we seek normal modes $U(t,x) = e^{\lambda t}V(x)$ of (2.14), which are

$$(\eta, u, q)(t,x) = e^{\lambda t}(\omega, v, r)(x). \tag{2.15}$$

We deduce the following system on (ω, v, r),

$$\begin{cases}\lambda\omega = v & \text{in } \Omega,\\ \lambda\rho_0 v - \mu\Delta v + \nabla r - \mathrm{m}^2\partial_3^2\omega + g\rho_0'\omega_3 e_3 = 0 & \text{in } \Omega,\\ \mathrm{div} v = 0 & \text{in } \Omega,\\ (r\mathrm{Id} - \mu(\nabla v + \nabla v^T))e_3 = \mathrm{m}^2\partial_3\omega + g\rho_+\omega_3 e_3 & \text{on } \Gamma_0,\\ v = 0 & \text{on } \Gamma_h.\end{cases} \tag{2.16}$$

That implies $\omega = \frac{1}{\lambda}v$ and

$$\begin{cases}\lambda^2\rho_0 v + \lambda\nabla r - \lambda\mu\Delta v - \mathrm{m}^2\partial_3^2 v = g\rho_0' v_3 e_3 & \text{in } \Omega,\\ \mathrm{div} v = 0 & \text{in } \Omega,\\ (\lambda r\mathrm{Id} - \lambda\mu(\nabla v + \nabla v^T))e_3 = \mathrm{m}^2\partial_3 v + g\rho_+ v_3 e_3 & \text{on } \Gamma_0,\\ v = 0 & \text{on } \Gamma_h.\end{cases} \tag{2.17}$$

Let $\mathbf{k} = (k_1, k_2) \in (L^{-1}\mathbb{Z}\backslash\{0\})^2$, we further assume that

$$\begin{cases}v_1(x) = \sin(k_1x_1 + k_2x_2)\psi(\mathbf{k}, x_3),\\ v_2(x) = \sin(k_1x_1 + k_2x_2)\varphi(\mathbf{k}, x_3),\\ v_3(x) = \cos(k_1x_1 + k_2x_2)\phi(\mathbf{k}, x_3),\\ r(x) = \cos(k_1x_1 + k_2x_2)\pi(\mathbf{k}, x_3).\end{cases} \tag{2.18}$$

Denote by $k = |\mathbf{k}| = \sqrt{k_1^2 + k_2^2}$, we deduce from (2.17) the system

$$\begin{cases}\lambda^2\rho_0\psi - \lambda k_1\pi + \lambda\mu(k^2\psi - \psi'') - \mathrm{m}^2\psi'' = 0 & \text{in } (-h,0),\\ \lambda^2\rho_0\varphi - \lambda k_2\pi + \lambda\mu(k^2\varphi - \varphi'') - \mathrm{m}^2\varphi'' = 0 & \text{in } (-h,0),\\ \lambda^2\rho_0\phi + \lambda\pi' + \lambda\mu(k^2\phi - \phi'') - \mathrm{m}^2\phi'' = g\rho_0'\phi & \text{in } (-h,0),\\ k_1\psi + k_2\varphi + \phi' = 0 & \text{in } (-h,0),\end{cases} \tag{2.19}$$

with the boundary conditions

$$\begin{cases} \lambda\mu(k_1\phi(0) - \psi'(0)) = \mathrm{m}^2\psi'(0), \\ \lambda\mu(k_2\phi(0) - \varphi'(0)) = \mathrm{m}^2\varphi'(0), \\ \lambda\pi(0) - g\rho_+\phi(0) - 2\lambda\mu\phi'(0) = \mathrm{m}^2\phi'(0), \\ \psi(-h) = \varphi(-h) = \phi(-h) = 0. \end{cases} \tag{2.20}$$

Note that

$$\pi = \frac{1}{k^2}\big(-\lambda\rho_0\phi' - \mu(k^2\phi - \phi'') + \frac{\mathrm{m}^2}{\lambda}\phi'''\big). \tag{2.21}$$

That implies the fourth-order ordinary differential equation

$$\lambda^2(k^2\rho_0\phi - (\rho_0\phi')') + \lambda\mu(\phi^{(4)} - 2k^2\phi'' + k^4\phi) = gk^2\rho_0'\phi - \mathrm{m}^2(\phi^{(4)} - k^2\phi''), \tag{2.22}$$

with the boundary conditions

$$\begin{cases} \phi(-h) = \phi'(-h) = 0, \\ \lambda\mu(\phi''(0) + k^2\phi(0)) = -\mathrm{m}^2\phi''(0), \\ \lambda\mu(\phi'''(0) - 3k^2\phi'(0)) + \mathrm{m}^2(\phi'''(0) - k^2\phi'(0)) = \lambda^2\rho_+\phi'(0) + gk^2\rho_+\phi(0). \end{cases} \tag{2.23}$$

The finding of normal modes of the form (2.15) to Eq. (2.14) relies on the investigation of the characteristic values $\lambda(k) \in \mathbb{C}$ ($\mathrm{Re}\lambda > 0$) as k fixed such that (2.22)-(2.23) has a nontrivial solution ϕ living at least in $H^4((-h,0))$.

Following [24, Lemma 2.1], we have that all characteristic values λ are real. Since our goal is to study the instability, we only consider positive λ and look for functions ϕ being real in what follows in the linear analysis.

Lemma 2.1. *For any $k > 0$, all characteristic values λ are always real. Let $L_0 := (\|\frac{\rho_0'}{\rho_0}\|_{L^\infty((-h,0))})^{-1}$ be the characteristic length of density profile, all characteristic values λ satisfy that $\lambda \leqslant \sqrt{\frac{g}{L_0}}$.*

As k is fixed, we state the following k-subcritical regime of magnetic field to investigate the existence of infinitely many characteristic values, thanks to the operator method initiated by Lafitte and Nguyễn [22]. We state our first theorem solving the ODE (2.22)–(2.23).

Theorem 2.1. *Let k be fixed and let ρ_0 satisfying (1.4)–(1.5). We define*

$$m_c(k) := \sqrt{\max_{\phi\in H^2_\star((-h,0))} \frac{gk^2\int_{-h}^0 \rho_0'\phi^2 - gk^2\rho_+\phi^2(0)}{\int_{-h}^0((\phi'')^2 + k^2(\phi')^2)}}. \tag{2.24}$$

Hence, let $\mathrm{c}_\star := \frac{\pi}{2h}(g\max_{(-h,0)}\rho_0')^{-1/2}$ and

$$0 < |m| < m_c(k) - \frac{1}{\mathrm{c}_\star k} \nearrow m_c \quad \text{as } k \to +\infty. \tag{2.25}$$

there exists an infinite sequence $(\lambda_n, \phi_n)_{n\geqslant 1}$ with $\lambda_n \in (0\sqrt{\frac{g}{L_0}})$ and $\phi_n \in H^\infty((-h,0))$ satisfying (2.22)–(2.23).

In view of (2.25), we prove the linear instability in the subcritical regime of magnetic field (1.6), showing the existence of infinitely many normal mode solutions to the linearized equations (2.14).

Theorem 2.2. *Let $|m| < m_c$. For some wave number $\mathbf{k} = (k_1,k_2) \in (L^{-1}\mathbb{Z})^2\backslash\{0\}$, there exists infinitely many normal mode solutions to the linearized equations (2.14).*

To close the linear section, we show that Λ defined by

$$0 < \Lambda := \sup_{\mathbf{k}\in(L^{-1}\mathbb{Z})^2\setminus\{0\}} \lambda_1(\mathbf{k}) \leqslant \sqrt{\frac{g}{L_0}}, \tag{2.26}$$

is the maximal growth rate of the linearized equations, see Proposition 3.6.

2.3. **Nonlinear instability.** The spectral analysis allows us to study fully nonlinear perturbation equations (2.13). To prove the nonlinear instability, we follow the procedure as in [24], explained below to make our paper self-contained.

In the first step, we construct the *a priori* energy estimates in low regularity regime. To do that, we introduce the following anisotropic Sobolev norm,

$$\|\cdot\|_{m,k,\Sigma} := \sum_{\alpha_1+\alpha_2\leqslant k} \|\partial_1^{\alpha_1}\partial_2^{\alpha_2}\cdot\|_{H^m(\Sigma)}.$$

Let us recall the perturbation terms $U = (\eta, u, q)$ and let $\varepsilon \in (0,1)$ be arbitrary, but fixed. We define the energy functional $\mathcal{E}(t) = \mathcal{E}(U(t)) > 0$ such that

$$\mathcal{E}(t) = \mathcal{E}_1(t) + \varepsilon\mathcal{E}_2(t), \tag{2.27}$$

where $\mathcal{E}_1, \mathcal{E}_2$ are given by

$$\begin{aligned}
\mathcal{E}_1(t) &:= \|\eta(t)\|_{1,4,\Omega} + \|\partial_3\eta(t)\|_{0,4,\Omega} + \|u(t)\|_{0,4,\Omega} + \|\partial_t u(t)\|_{0,2,\Omega},\\
\mathcal{E}_2(t) &:= \|\eta(t)\|_{H^5(\Omega)} + \sum_{j=0}^{2}\left(\|\partial_t^j u(t)\|^2_{H^{4-2j}(\Omega)} + \|\partial_t^j\partial_3\eta(t)\|_{L^2(\Omega)}\right)\\
&\qquad + \|q(t)\|_{H^3(\Omega)} + \|\partial_t q(t)\|_{H^1(\Omega)}.
\end{aligned}$$

We also define the dissipation term $\mathcal{D}(t) = \mathcal{D}(U(t)) > 0$ such that

$$\mathcal{D}(t) = \mathcal{D}_1(t) + \varepsilon\mathcal{D}_2(t), \tag{2.28}$$

where $\mathcal{D}_1, \mathcal{D}_2$ are given by

$$\begin{aligned}
\mathcal{D}_1(t) &= \|\partial_3\eta(t)\|_{0,4,\Omega} + \|u(t)\|_{1,4,\Omega} + \|\partial_t u(t)\|_{0,2,\Omega},\\
\mathcal{D}_2(t) &= \|\eta(t)\|_{H^5(\Omega)} + \|u(t)\|^2_{H^5(\Omega)} + \|\partial_t u(t)\|^2_{H^3(\Omega)} + \|\partial_t^2 u(t)\|^2_{H^1(\Omega)}\\
&\qquad + \|q(t)\|^2_{H^4(\Omega)} + \|\partial_t q(t)\|^2_{H^2(\Omega)}.
\end{aligned}$$

The local existence of regular solution to (2.13) follows from [7, Theorem 6.3]. That means, there exists $\delta_0 > 0$ sufficiently small such that for any $\delta \in (0, \delta_0)$, Eq. (2.13) with the initial data (η_0, u_0) satisfying the appropriate compatibility conditions and $\mathcal{E}_1(0) + \mathcal{E}_2(0) \leqslant \delta$, has a unique solution (η, u, q) existing on the time interval $[0, T_{\max})$ and η is a C^2-diffeomorphism for each $t \in [0, T_{\max})$. With that regular solution (η, u, q) of (2.13) on a finite time interval $[0, T_{\max})$, we aim at showing the *a priori* energy estimates for the nonlinear equations (2.13).

Proposition 2.1. *Let $\varepsilon \in (0,1)$ be arbitrary, but fixed. If $\sup_{0\leqslant s\leqslant t}(\mathcal{E}_1 + \mathcal{E}_2)(s) \ll 1$, there exists $C_0 > 0$ independent of ε such that the following inequality holds*

$$\begin{aligned}
\mathcal{E}^2(t) + \int_0^t \mathcal{D}^2(s)ds &\leqslant C_\varepsilon\mathcal{E}^2(0) + C_\varepsilon\mathcal{E}^3(t) + C_0\varepsilon\int_0^t \mathcal{E}^2(s)ds\\
&\qquad + C_0\varepsilon^{-8}\int_0^t \|(\eta,u)(s)\|^2_{L^2(\Omega)}ds + C_\varepsilon\int_0^t \mathcal{E}(\mathcal{E}^2 + \mathcal{D}^2)(s)ds.
\end{aligned} \tag{2.29}$$

In the second step, in view of getting infinitely many characteristic values of the linearized equations (2.14), we formulate appropriate initial data to Eq. (2.13). Thanks to (2.26), we define the non-empty set

$$S_\Lambda := \left\{ \mathbf{k} \in (L^{-1}\mathbb{Z})^2 \backslash \{0\} : \lambda_1(\mathbf{k}) > \frac{2\Lambda}{3} \right\}.$$

We further fix a $\mathbf{k} \in S_\Lambda$. Hence, there is a unique $\mathsf{P} \in \mathbb{N}^\star$ such that

$$\Lambda \geqslant \lambda_1(\mathbf{k}) > \lambda_2(\mathbf{k}) > \cdots > \lambda_{\mathsf{P}}(\mathbf{k}) > \frac{2\Lambda}{3} > \lambda_{\mathsf{P}+1}(\mathbf{k}) > \dots . \tag{2.30}$$

In view of getting infinitely many characteristic values of the linearized problem, we consider a linear combination of normal modes

$$U^{\mathsf{N}}(t,x) = \sum_{j=1}^{\mathsf{N}} \mathsf{c}_j e^{\lambda_j t} V_j(x) \quad \text{(for any natural number } \mathsf{N}) \tag{2.31}$$

to be an approximate solution to the nonlinear equations (2.13), with constants c_j being chosen such that

$$\text{at least one of } \mathsf{c}_j \ (1 \leqslant j \leqslant \mathsf{N}) \text{ is non-zero} \tag{2.32}$$

and

$$\frac{1}{2}|\mathsf{c}_{j_m}|\|u_{j_m}\|_{L^2(\Omega)} > \sum_{j \geqslant j_m+1} |\mathsf{c}_j|\|u_j\|_{L^2(\Omega)} \quad (j_m := \min\{j : 1 \leqslant j \leqslant \mathsf{N}, \mathsf{c}_j \neq 0\}). \tag{2.33}$$

We would like to use $U^{\mathsf{N}}(0,x)$ as the initial data for the nonlinear equations (2.13). Unfortunately, $U^{\mathsf{N}}(0,x)$ does not satisfy the compatibility conditions in general due to the incompressibility of the linearized equations. Hence, using an abstract argument from [9, Section 5C], which was also used in [32, 31], we obtain the modified initial data $U^{\mathsf{N}}(0,x)$.

Proposition 2.2. *There exist a number $\delta_0 > 0$ and a family of initial data*

$$U_0^{\delta,\mathsf{N}}(x) = \delta U^{\mathsf{N}}(0,x) + \delta^2 U_\star^{\delta,\mathsf{N}}(x) \tag{2.34}$$

for $\delta \in (0, \delta_0)$ such that

(1) *$\mathcal{E}(U_\star^{\delta,\mathsf{N}}(t)) \leqslant C_{\mathsf{N}}^\star < \infty$, with $C_{\mathsf{N}}^\star$ being independent of δ,*
(2) *$U_0^{\delta,\mathsf{N}}$ satisfies the nonlinear compatibility conditions required for a solution $U^{\delta,\mathsf{N}}$ to the nonlinear problem (2.13) to exist in the norm $\|\cdot\|_{\mathcal{E}} := \mathcal{E}(\cdot)$.*

In the third step, with the solution $U^{\delta,\mathsf{N}}$, we now define the difference function

$$U^d = U^{\delta,\mathsf{N}} - \delta U^{\mathsf{N}}.$$

Since $U^{\delta,\mathsf{N}}$ solves the nonlinear equations (2.13) and U^{N} solves the linearized equations (2.14), we obtain that U^d is a solution to the nonlinear equations

$$\begin{cases} \partial_t \eta^d = u^d & \text{in } \Omega, \\ \rho_0 \partial_t u^d - \mu \Delta u^d + \nabla q^d - \mathrm{m}^2 \partial_3^2 \eta^d - g\rho_0' \eta_3^d e_3 = \mathcal{Q}_1(U^d) & \text{in } \Omega, \\ \mathrm{div} u^d = \mathcal{Q}_2(U^d) & \text{in } \Omega, \\ (q^d \mathrm{Id} - \mu \mathbb{S}(u^d)) e_3 = \mathrm{m}^2 \partial_3 \eta^d + g\rho_+ \eta_3^d e_3 + \mathcal{Q}_3(U^d) & \text{on } \Gamma_0, \\ u^d = 0 & \text{on } \Gamma_h, \end{cases} \tag{2.35}$$

with the initial data

$$U^d(0) = (\eta^d, u^d, q^d)(0) = \delta^2 U_\star^{\delta,\mathsf{N}}. \tag{2.36}$$

For t small enough, we deduce the following bound in time (see Proposition 5.1),

$$\|(\zeta^d, u^d)(t)\|_{L^2(\Omega)}^2 \lesssim \delta^3 \Big(\sum_{j=j_m}^{\mathsf{N}} |\mathsf{c}_j| e^{\lambda_j t} + \max(0, \mathsf{N} - \mathsf{P}) \max_{\mathsf{P}+1 \leqslant j \leqslant \mathsf{N}} |\mathsf{c}_j| e^{\frac{2}{3}\Lambda t} \Big)^3,$$

That relies on some energy estimates of Eq. (2.35) and the bound in time of a suitable Sobolev norm of $U^{\delta,M}(t)$ (see Lemma 5.1), which we obtain thanks to the *a priori* energy estimate (2.29). Combining those estimates, we obtain the following nonlinear instability result.

Theorem 2.3. *Let ρ_0 satisfy (1.4)–(1.5) and let $|m| < m_c$. Let N be an arbitrary integer, there exist two positive constants ν_0, δ_0 sufficiently small and another constant $m_0 > 0$, so that for any $\delta \in (0, \delta_0)$ the nonlinear equations (2.13) with the initial data (2.34), i.e.*

$$\delta \sum_{j=1}^{\mathsf{N}} \mathsf{c}_j V_j(x) + \delta^2 U_\star^{\delta,\mathsf{N}}(x),$$

satisfying (2.32)-(2.33) admits a unique local strong solution $U^{\delta,\mathsf{N}}$ such that

$$\|u^{\delta,\mathsf{N}}(T^\delta)\|_{L^2(\Omega)} \geqslant m_0\nu_0, \tag{2.37}$$

where $T^\delta \in (0, T_{\max})$ is given by

$$\delta \sum_{j=1}^{\mathsf{N}} |\mathsf{c}_j| e^{\lambda_j T^\delta} = \nu_0.$$

3. The linear instability

3.1. Auxillary operators.

We begin with some useful operators.

Proposition 3.1. *Let us define the function space*

$$H_\star^2((-h,0)) = \{\vartheta \in H_\star^2(-h,0) : \vartheta(-h) = \vartheta'(-h) = 0\},$$

and the bilinear form on $H_\star^2((-h,0))$

$$\begin{aligned} \mathscr{B}_{k,\lambda}(\vartheta,\varrho) = \lambda \int_{-h}^0 \rho_0(k^2\vartheta\varrho + \vartheta'\varrho') + \mu \int_{-h}^0 \big((\vartheta'' + k^2\vartheta)(\varrho'' + k^2\varrho) + 4k^2\vartheta'\varrho'\big) \\ + \frac{m^2}{\lambda}\int_{-h}^0 (\vartheta''\varrho'' + k^2\vartheta'\varrho') + \frac{gk^2\rho_+}{\lambda}\vartheta(0)\varrho(0). \end{aligned} \tag{3.1}$$

Let $(H_\star^2((-h,0)))'$ be the dual space of $H_\star^2((-h,0))$, which is associated with the norm $\sqrt{\mathscr{B}_{k,\lambda}(\cdot,\cdot)}$, there exists a unique operator

$$Y_{k,\lambda} \in \mathcal{L}(H_\star^2((-h,0)), (H_\star^2((-h,0)))'),$$

that is also bijective, such that for all $\vartheta, \varrho \in H_\star^2((-h,0))$,

$$\mathscr{B}_{k,\lambda}(\vartheta,\varrho) = \langle Y_{k,\lambda}\vartheta, \varrho\rangle. \tag{3.2}$$

The proof of Proposition 3.1 is straightforward thanks to Riesz's representation theorem, hence we omit details. The next proposition is to devoted to studying the properties of $Y_{k,\lambda}$.

Proposition 3.2. *For all $\vartheta \in H_\star^2((-h,0))$, we have*

$$Y_{k,\lambda}\vartheta = \lambda(k^2\rho_0\vartheta - (\rho_0\vartheta')') + \mu(\vartheta^{(4)} - 2k^2\vartheta'' + k^4\vartheta) + \frac{m^2}{\lambda}(\vartheta^{(4)} - k^2\vartheta'') \quad \text{in } \mathcal{D}'((-h,0)).$$

Let $f \in L^2((-h,0))$ be given, there exists a unique $\vartheta \in H_\star^2((-h,0))$ such that

$$Y_{k,\lambda}\vartheta = f \text{ in } (H_\star^2((-h,0)))'. \tag{3.3}$$

Moreover, $\vartheta \in H^4((-h,0))$ and satisfies the boundary conditions (2.23).

The proof of Proposition 3.2 is due to a bootstrap argument, which is followed by [22, Proposition 3.3]. Hence we refer the details to [25, Proposition 3.3]. We have the following proposition on $Y_{k,\lambda}^{-1}$.

Proposition 3.3. *The operator $Y_{k,\lambda}^{-1} : L^2((-h,0)) \to L^2((-h,0))$ is compact and self-hdjoint.*

We prove Proposition 3.2 thanks to the continuous injection from $H^4((-h,0))$ to $L^2((-h,0))$, in the same line of [22, Proposition 3.4].

Let $\mathcal{M}$ be the multiplication by $\sqrt{\rho_0'}$. We now study the operator $S_{a,k,\lambda} := \mathcal{M}Y_{k,\lambda}^{-1}\mathcal{M}$. Owing to Proposition 3.3, we obtain the following.

Proposition 3.4. *The operator $S_{k,\lambda} : L^2((-h,0)) \to L^2((-h,0))$ is compact and self-adjoint.*

3.2. **A sequence of characteristic values.** As a result of the spectral theory of compact and self-adjoint operators, the point spectrum of $S_{k,\lambda}$ is discrete, i.e. is a positive sequence $\{\gamma_n(\lambda,k)\}_{n\geqslant 1}$ of eigenvalues of $S_{k,\lambda}$ decreasing towards 0 as $n \to \infty$, associated with normalized orthogonal eigenfunctions $\{\varpi_n\}_{n\geqslant 1}$ in $L^2((-h,0))$. That means

$$\gamma_n(\lambda,k)\varpi_n = S_{k,\lambda}\varpi_n = \mathcal{M}Y_{k,\lambda}^{-1}\mathcal{M}\varpi_n.$$

So that with $\phi_n = Y_{k,\lambda}^{-1}\mathcal{M}\varpi_n \in H^4((-h,0))$ satisfying (2.23), one has

$$\gamma_n(\lambda,k)Y_{k,\lambda}\phi_n = \rho_0'\phi_n. \tag{3.4}$$

In order to verify that ϕ_n is a solution of (2.22)-(2.23), we are left to look for real values of λ_n satisfying

$$\gamma_n(\lambda,k) = \frac{\lambda}{gk^2}. \tag{3.5}$$

To solve (3.5), we need the three following lemmas.

Lemma 3.1. *There holds*

$$\max_{\theta\in H^1((-h,0)),\theta(-h)=0} \frac{k^2\int_{-h}^0 \theta^2}{\int_{-h}^0 (\theta')^2} = \frac{4h^2}{\pi^2}. \tag{3.6}$$

The proof of Lemma 3.1 is due to Lagrangian multiplier method. Hence, we omit the details here.

Lemma 3.2. *For each n, $\gamma_n(\lambda,k)$ and ϕ_n are differentiable in λ.*

The proof of Lemma 3.2 is the same as [22, Lemma 3.2], we omit the details here.

Lemma 3.3. *For each n, $\gamma_n(\lambda,k)$ is strictly decreasing in $\lambda > 0$.*

Proof. Let $z_n = \frac{d\phi_n}{d\lambda}$, it follows from (3.4) that

$$Y_{k,\lambda}z_n + k^2\rho_0\phi_n - (\rho_0\phi_n')' - \frac{\mathrm{m}^2}{\lambda^2}(\phi_n^{(4)} - k^2\phi_n'') = \frac{1}{\gamma_n}\rho_0'z_n + \frac{d}{d\lambda}\Big(\frac{1}{\gamma_n}\Big)\rho_0'\phi_n.$$

That implies

$$\begin{aligned}\int_{-h}^0 (k^2\rho_0\phi_n - (\rho_0\phi_n')')\phi_n - \frac{\mathrm{m}^2}{\lambda^2}\int_{-h}^0 (\phi_n^{(4)} - k^2\phi_n'')\phi_n + \int_{-h}^0 (Y_{k,\lambda}z_n)\phi_n \\ = \frac{1}{\gamma_n}\int_{-h}^0 \rho_0'z_n\phi_n + \frac{d}{d\lambda}\Big(\frac{1}{\gamma_n}\Big)\int_{-h}^0 \rho_0'\phi_n^2.\end{aligned} \tag{3.7}$$

Note that

$$\begin{cases} z_n(-h) = z_n'(-h) = 0, \\ \lambda\mu(z_n''(0) + k^2z_n(0)) + \mathrm{m}^2z_n''(0) = -\mu(\phi_n''(0) + k^2\phi_n(0)), \\ \lambda\mu(z_n'''(0) - 3k^2z_n'(0)) + \mathrm{m}^2(z_n'''(0) - k^2z_n'(0)) - \lambda^2\rho_+z_n'(0) - gk^2\rho_+z_n(0) \\ \qquad = 2\lambda\rho_+\phi_n'(0) - \mu(\phi_n'''(0) - 3k^2\phi_n'(0)). \end{cases} \tag{3.8}$$

Using the integration by parts and (2.23)-(3.8), we have

$$
\begin{aligned}
\int_{-h}^{0}(Y_{k,\lambda}z_n)\phi_n &= \int_{-h}^{0}(Y_{k,\lambda}\phi_n)z_n + \Big(\mu(z_n''' - 3k^2 z_n')\phi_n - \mu(z_n'' + k^2 z_n)\phi_n' - \lambda\rho_0 z_n'\phi_n\Big)(0)\\
&\quad - \Big(\mu(\phi_n''' - 3k^2\phi_n')z_n - \mu(\phi_n'' + k^2\phi_n)\phi_n' - \lambda\rho_0\phi_n' z_n\Big)(0)\\
&\quad + \frac{\mathrm{m}^2}{\lambda}\big((z_n'' - k^2 z_n)\phi_n - z_n''\phi_n'\big)(0) - \frac{\mathrm{m}^2}{\lambda}\big((\phi_n''' - k^2\phi_n')z_n - \phi_n'' z_n'\big)(0)\\
&= \frac{1}{\gamma_n}\int_{-h}^{0}\rho_0' z_n\phi_n + 2\rho_+\phi_n'(0)\phi_n(0) - \frac{\mu}{\lambda}(\phi_n'''(0) - 3k^2\phi_n'(0))\phi_n(0)\\
&\quad + \frac{\mu}{\lambda}(\phi_n''(0) + k^2\phi_n(0))\phi_n'(0).
\end{aligned}
$$

In the same manner, we have

$$
\begin{aligned}
&\int_{-h}^{0}(k^2\rho_0\phi_n - (\rho_0\phi_n')')\phi_n - \frac{\mathrm{m}^2}{\lambda^2}\int_{-h}^{0}(\phi_n^{(4)} - k^2\phi_n'')\phi_n\\
&= \int_{-h}^{0}\rho_0(k^2\phi_n^2 + (\phi_n')^2) - \rho_+\phi_n(0)\phi_n'(0) - \frac{\mathrm{m}^2}{\lambda^2}\int_{-h}^{0}(k^2(\phi_n')^2 + (\phi_n'')^2)\\
&\quad - \frac{\mathrm{m}^2}{\lambda^2}\big((\phi_n'''(0) - k^2\phi_n'(0))\phi_n(0) - \phi_n''(0)\phi_n'(0)\big)
\end{aligned}
$$

Combining those above integrals, we deduce

$$
\begin{aligned}
&\int_{-h}^{0}(k^2\rho_0\phi_n - (\rho_0\phi_n')')\phi_n - \frac{\mathrm{m}^2}{\lambda^2}\int_{-h}^{0}(\phi_n^{(4)} - k^2\phi_n'')\phi_n + \int_{-h}^{0}(Y_{k,\lambda}z_n)\phi_n\\
&= \rho_+\phi_n'(0)\phi_n(0) - \frac{\mu}{\lambda}(\phi_n'''(0) - 3k^2\phi_n'(0))\phi_n(0) + \frac{\mu}{\lambda}(\phi_n''(0) + k^2\phi_n(0))\phi_n'(0)\\
&\quad - \frac{\mathrm{m}^2}{\lambda^2}\big((\phi_n'''(0) - k^2\phi_n'(0))\phi_n(0) - \phi_n''(0)\phi_n'(0)\big)\\
&\quad + \int_{-h}^{0}\rho_0(k^2\phi_n^2 + (\phi_n')^2) - \frac{\mathrm{m}^2}{\lambda^2}\int_{-h}^{0}(k^2(\phi_n')^2 + (\phi_n'')^2)\\
&= -\frac{gk^2\rho_+}{\lambda^2}\phi_n^2(0) + \int_{-h}^{0}\rho_0(k^2\phi_n^2 + (\phi_n')^2) - \frac{\mathrm{m}^2}{\lambda^2}\int_{-h}^{0}(k^2(\phi_n')^2 + (\phi_n'')^2).
\end{aligned}
$$

From the definition of $\mathrm{m}_c(k)$, we get further

$$
\begin{aligned}
\lambda^2\frac{d}{d\lambda}\Big(\frac{1}{\gamma_n}\Big)\int_{-h}^{0}\rho_0'\phi_n^2 &= \lambda^2\int_{-h}^{0}\rho_0(k^2\phi_n^2 + (\phi_n')^2) - \Big(gk^2\rho_+\phi_n^2(0) + \mathrm{m}^2\int_{-h}^{0}((\phi_n'')^2 + k^2(\phi_n')^2)\Big)\\
&\geqslant \lambda^2\int_{-h}^{0}\rho_0(k^2\phi_n^2 + (\phi_n')^2) + (\mathrm{m}_c^2(k) - \mathrm{m}^2)\int_{-h}^{0}((\phi_n'')^2 + k^2(\phi_n')^2)\\
&\quad - gk^2\int_{-h}^{0}\rho_0'\phi_n^2.
\end{aligned}
$$

Now, using Lemma 3.1, we obtain

$$
\int_{-h}^{0}((\phi_n'')^2 + k^2(\phi_n')^2) \geqslant k^2\int_{-h}^{0}(\phi_n')^2 \geqslant \frac{\pi^2k^4}{4h^2}\int_{-h}^{0}\phi_n^2 \geqslant \mathrm{c}_\star^2\int_{-h}^{0}gk^2\rho_0'\phi_n^2.
$$

Consequently,

$$
\lambda^2\frac{d}{d\lambda}\Big(\frac{1}{\gamma_n}\Big)\int_{-h}^{0}\rho_0'\phi_n^2 > \big(\mathrm{c}_\star^2(\mathrm{m}_c^2(k) - \mathrm{m}^2)k^2 - 1\big)\int_{-h}^{0}gk^2\rho_0'\phi_n^2.
$$

For m satisfying (2.25), we deduce that

$$\lambda^2 \frac{d}{d\lambda}\Big(\frac{1}{\gamma_n}\Big) \int_{-h}^0 \rho_0' \phi_n^2 > 0.$$

That implies γ_n is a decreasing function in $\lambda > 0$. □

Now we are in position to solve (3.5).

Proposition 3.5. *For each $n \geqslant 1$, there exists a unique $\lambda_n > 0$ solving (3.5). In addition, λ_n decreases towards 0 as n goes to ∞.*

Proof. Using (3.4), we know that

$$\frac{1}{\gamma_n(\lambda, k)} \int_{-h}^0 \rho_0' \phi_n^2 dx_3 = \int_{-a}^0 (Y_{k,\lambda}\phi_n)\phi_n dx_3 = \mathscr{B}_{k,\lambda}(\phi_n, \phi_n),$$

Keep in mind (3.1), we deduce that

$$\frac{1}{\gamma_n(\lambda, k)} \int_{-h}^0 \rho_0' \phi_n^2 dx_3 \geqslant \lambda \int_{-h}^0 k^2 \rho_0 \phi_n^2 dx_3 + \mu \int_{-h}^0 k^4 \phi_n^2 dx_3,$$

that implies

$$\frac{1}{L_0 \gamma_n(\lambda, k)} \geqslant \lambda k^2 + \frac{\mu k^4}{\rho_+}.$$

Hence

$$\lim_{\lambda \to \sqrt{\frac{g}{L_0}}} \frac{\lambda}{\gamma_n(\lambda, k)} > g k^2. \tag{3.9}$$

Since $\gamma_n(\lambda, k)$ is a decreasing function, we have that $\gamma_n(\lambda, k) \geqslant \gamma_n(\frac{1}{2}\sqrt{\frac{g}{L_0}}, k)$ for all $\lambda \leqslant \frac{1}{2}\sqrt{\frac{g}{L_0}}$. It yields,

$$\lim_{\lambda \to 0} \frac{\lambda}{\gamma_n(\lambda, k)} \leqslant \lim_{\lambda \to 0} \frac{\lambda}{\gamma_n(\frac{1}{2}\sqrt{\frac{g}{L_0}}, k)} = 0. \tag{3.10}$$

Combining (3.9), (3.10) and the fact that γ_n is decreasing in λ, we obtain a unique λ_n solving (3.5).

We prove that the sequence $(\lambda_n)_{n\geqslant 1}$ is decreasing. Indeed, if $\lambda_m < \lambda_{m+1}$ for some $m \geqslant 1$, we have $\gamma_m(\lambda_m, k) > \gamma_m(\lambda_{m+1}, k)$. Meanwhile, we also have $\gamma_m(\lambda_{m+1}, k) > \gamma_{m+1}(\lambda_{m+1}, k)$. That implies

$$\frac{\lambda_m}{g k^2} = \gamma_m(\lambda_m, k) > \gamma_{m+1}(\lambda_{m+1}, k) = \frac{\lambda_{m+1}}{g k^2}.$$

That contradiction tells us that $(\lambda_n)_{n\geqslant 1}$ is a decreasing sequence.

To conclude Proposition 3.5, we prove that $\lim_{n\to\infty} \lambda_n = 0$. Indeed, suppose that $\lim_{n\to\infty} \lambda_n = c_0 > 0$, one has that $\lambda_n \geqslant c_0$ for all $n \geqslant 1$. This yields

$$\gamma_n(c_0, k) \geqslant \gamma_n(\lambda_n, k) = \frac{\lambda_n}{g k^2} \geqslant \frac{c_0}{g k^2}.$$

Letting $n \to \infty$, we obtain that $0 \geqslant \frac{c_0}{g k^2}$, which is a contradiction. Hence, $\lim_{n\to\infty} \lambda_n = 0$. Proposition 3.5 is proven. □

3.3. **Proof of Theorems 2.1, 2.2.** Thanks to Proposition 3.5, we are able to prove our linear result.

Proof of Theorem 2.1. For each λ_n being found in Proposition 3.5, let $\phi_n = Y_{k,\lambda_n}^{-1}\mathcal{M}w_n(x_3)$. Therefore, the function $\phi_n \in H^4((-h,0))$ is a solution of (2.22) satisfying (2.23) as $\lambda = \lambda_n$ for each $n \geqslant 1$. Using a bootstrap argument, we have $\phi_n \in H^\infty((-h,0))$. Proof of Theorem 2.1 is complete. □

Once we get infinite many solutions $(\lambda_n,\phi_n)_{n\geqslant 1}$ to (2.22)-(2.23), we go back to the linearized equations (2.14).

Proof of Theorem 2.2. For each solution $\lambda_n \in (0,\sqrt{\frac{g}{L_0}})$ of (3.5), we have a solution ϕ_n in $H^4((-h,0))$ of (2.22) as $\lambda = \lambda_n$, being found in Theorem 2.1. Furthermore, $\phi_n \in H^\infty((-h,0))$. We find uniquely $\pi_n \in H^\infty((-h,0))$ from (2.21) such that

$$\pi_n(\mathbf{k},x_3) = \frac{1}{k^2}(-\lambda_n\rho_0\phi_n' - \mu(k^2\phi_n' - \phi_n''') + \frac{\mathrm{m}^2}{\lambda}\phi_n''')(\mathbf{k},x_3).$$

To look for ψ_n, we rewrite (2.19) as a second order ODE,

$$-\mu\psi_n'' + (\lambda_n\rho_0\psi_n + \mu k^2\psi_n - k_1\pi_n) = 0.$$

By the ODE theory on a bounded interval, we obtain a unique solution $\psi_n \in H^\infty((-h,0))$, where the solution ψ_n depends on the known functions ϕ_n and π_n. We get φ_n in a similar way. Hence, $(\psi_n,\varphi_n,\phi_n,\pi_n) \in (H^\infty((-h,0)))^4$ is a solution of (2.19)-(2.20).

Following (2.18), we now construct the functions

$$\begin{aligned}
v_{1,n}(\mathbf{k},x) &= \sin(k_1x_1+k_2x_2)\psi_n(\mathbf{k},x_3),\\
v_{2,n}(\mathbf{k},x) &= \sin(k_1x_1+k_2x_2)\varphi_n(\mathbf{k},x_3),\\
v_{3,n}(\mathbf{k},x) &= \cos(k_1x_1+k_2x_2)\phi_n(\mathbf{k},x_3),\\
r_n(\mathbf{k},x_3) &= \cos(k_1x_1+k_2x_2)\pi_n(\mathbf{k},x_3),\\
\omega_n(\mathbf{k},x) &= \frac{1}{\lambda_n}v_n(\mathbf{k},x).
\end{aligned}$$

Hence, for each $n \geqslant 1$,

$$(\eta_n(t,\mathbf{k},x), u_n(t,\mathbf{k},x), q_n(t,\mathbf{k},x)) = e^{\lambda_n(\mathbf{k})t}(\omega_n,v_n,r_n)(\mathbf{k},x)$$

is a real-valued and smooth solution to (2.14). The proof of Theorem 2.2 is finished. □

3.4. **Maximal growth rate.** We state the following property on the largest characteristic value λ_1 found in Theorem 2.2, whose proof is followed by using the self-adjointness of $S_{k,\lambda}$ and is similar to [24, Lemma 4.2].

Lemma 3.4. *Let us recall the bilinear form $\mathscr{B}_{k,\lambda}$ on $H^2((-h,0))$ (3.1) and (λ_1,ϕ_1) from Theorem 2.2. We have that*

$$\frac{1}{gk^2} = \max_{\phi\in H^2((-h,0))}\frac{\int_{-h}^0\rho_0'\phi^2}{\lambda_1\mathscr{B}_{k,\lambda_1}(\phi,\phi)}, \tag{3.11}$$

and the extremal problem (3.11) is attained by ϕ_1 restricted on $(-h,0)$ up to a constant.

Taking horizontal Fourier transform and using (3.11) and the definition of Λ (2.26), we follow [32, Lemma 2.6] or [25, Lemma 3.7] to get the following inequality.

Lemma 3.5. *For any u such that $\mathrm{div}u = 0$, there holds*

$$\int_\Omega g\rho_0'|u_3|^2 - m^2\int_\Omega|\partial_3u|^2 \leqslant \int_{\Gamma_0}g\rho_+|u_3|^2 + \Lambda^2\int_\Omega\rho_0|u|^2 + \frac{1}{2}\Lambda\int_\Omega\mu|\mathbb{S}u|^2. \tag{3.12}$$

Thanks to Lemmas 3.4, 3.5, we are able to get that Λ is the maximal growth rate of the linearized equations (2.14) in the following sense:

Proposition 3.6. *For arbitrary solution (ζ, u, η) of the linearized equations (2.14), the following inequality holds*

$$\|(\eta,u)(t)\|_{H^1}^2+\|\partial_t u(t)\|_{L^2(\Omega)}^2+\int_0^t\|u(s)\|_{H^1(\Omega)}^2ds\leqslant C(\|\partial_t u(0)\|_{L^2(\Omega)}^2+\|u(0)\|_{H^1(\Omega)}^2)e^{2\Lambda t},\quad(3.13)$$

for some universal constant $C>0$.

The line of proving (3.13) is as same as that one in [32, Theorem 2.7], [11, Proposition 5.2] or [25, Proposition 3.8]. Hence, we omit details.

4. A PRIORI ENERGY ESTIMATES

We employ the Einstein convention of summing over repeated indices. Throughout this section, we will employ the notation $a\lesssim b$ to mean that $a\leqslant Cb$ for a universal constant $C>0$ independent. When a constant C depends on ε, we will write $C=C_\varepsilon$ or $a\lesssim_\varepsilon b$.

4.1. **Temporal estimates.** In this section, we establish the temporal estimates for the velocity. Applying the temporal differential operator ∂_t^j $(j\geqslant 0)$ to (2.12), the resulting equations are

$$\begin{cases}\partial_t^{j+1}\eta=\partial_t^j u & \text{in } \Omega,\\ \rho_0\partial_t^{j+1}u+\nabla_{\mathcal{A}}\partial_t^j q-\mu\Delta_{\mathcal{A}}\partial_t^j u-\mathrm{m}^2\partial_3^2\partial_t^j\eta-g\rho_0'\partial_t^j\eta_3 e_3=F^{1,j} & \text{in } \Omega,\\ \mathrm{div}_{\mathcal{A}}\partial_t^j u=F^{2,j} & \text{in } \Omega,\\ (\partial_t^j q\mathrm{Id}-\mu\mathbb{S}_{\mathcal{A}}\partial_t^j u)\mathcal{N}=g\rho_+\partial_t^j\eta_3\mathcal{N}+F^{3,j} & \text{on } \Gamma_0,\\ \partial_t^j u=0 & \text{on } \Gamma_h,\end{cases}\quad(4.1)$$

where

$$\begin{aligned}F_i^{1,j}&=\partial_t^j(\mathcal{Q}_p+\mathcal{Q}_g)_i+\sum_{0<l\leqslant j}C_j^l\mu(\mathcal{A}_{lk}\partial_k(\partial_t^l\mathcal{A}_{lm}\partial_t^{j-l}\partial_m u_i)+\partial_t^l\mathcal{A}_{lk}\partial_t^{j-l}\partial_k(\mathcal{A}_{lm}\partial_m u_i))\\ &\quad-\sum_{0<l\leqslant j}\partial_t^l\mathcal{A}_{ik}\partial_t^{j-l}\partial_k q,\\ F^{2,j}&=-\sum_{0<l\leqslant j}C_j^l\partial_t^l\mathcal{A}_{ik}\partial_k(\partial_t^{j-l}u_i),\\ F^{3,j}&=\mu\sum_{0<l\leqslant j}C_j^l(\partial_t^l(\mathcal{A}_{ik}\mathcal{N}_m)\partial_k\partial_t^{j-l}u_m+\partial_t^l(\mathcal{A}_{mk}\mathcal{N}_m)\partial_k\partial_t^{j-l}u_i)+\sum_{0<l\leqslant j}C_j^l\partial_t^l\mathcal{N}_i\partial_t^{j-l}(g\rho_+\eta-q).\end{aligned}$$

Proposition 4.1. *For $j=0$ or 1, the following inequality holds*

$$\begin{aligned}&\|\partial_t^j u(t)\|_{L^2(\Omega)}^2+\|\partial_t^j\partial_3\eta(t)\|_{L^2(\Omega)}^2+\int_0^t\|\partial_t^j u(s)\|_{H^1(\Omega)}^2ds\\ &\quad\leqslant C_\varepsilon\mathcal{E}^2(0)+C\int_0^t\|(\eta_3,u_3)(s)\|_{L^2(\Omega)}^2ds+C_\varepsilon\int_0^t\mathcal{E}(\mathcal{E}^2+\mathcal{D}^2)(s)ds.\end{aligned}\quad(4.2)$$

We also have

$$\begin{aligned}&\|\partial_t^2 u(t)\|_{L^2(\Omega)}^2+\|\partial_t^2\partial_3\eta(t)\|_{L^2(\Omega)}^2+\int_0^t\|\partial_t^2 u(s)\|_{H^1(\Omega)}^2ds\\ &\quad\leqslant C_\varepsilon\mathcal{E}^2(0)+C_\varepsilon\mathcal{E}^3(t)+C\int_0^t\|(\eta_3,u_3)(s)\|_{L^2(\Omega)}^2+C_\varepsilon\int_0^t\mathcal{E}(\mathcal{E}^2+\mathcal{D}^2)(s)ds.\end{aligned}\quad(4.3)$$

To prove Proposition 4.1, we need the two following lemmas.

Lemma 4.1. *For any $j \geqslant 0$, one has*

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\Big(\int_\Omega \rho_0|\partial_t^j u|^2 + m^2\int_\Omega |\partial_3\partial_t^j\eta|^2 + \int_{\Gamma_0} g\rho_+|\partial_t^j\eta_3|^2\Big) + \frac{1}{2}\int_\Omega \mu|\mathbb{S}_{\mathcal{A}}\partial_t^j u|^2 \\ &= \int_\Omega g\rho_0'\partial_t^j\eta_3\partial_t^j u_3 + \int_\Omega F^{1,j}\cdot\partial_t^j u + \int_\Omega F^{2,j}\cdot\partial_t^j q - \int_{\Gamma_0} F^{3,j}\cdot\partial_t^j u - \int_{\Gamma_0} g\rho_+\partial_t^j\eta_3\partial_t^j u\cdot(\mathcal{N}-e_3).\end{aligned} \tag{4.4}$$

Proof. For $j \geqslant 0$, we multiply $(4.1)_2$ by $\partial_t^j u$ to get that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega \rho_0|\partial_t^j u|^2 + \int_\Omega(\nabla_{\mathcal{A}}\partial_t^j q - \mu\Delta_{\mathcal{A}}\partial_t^j u)\cdot\partial_t^j u \\ - \mathrm{m}^2\int_\Omega \partial_3^2\partial_t^j\eta\cdot\partial_t^j u - \int_\Omega g\rho_0'\partial_t^j\eta_3\partial_t^j u_3 = \int_\Omega F^{1,j}\cdot\partial_t^j u.\end{aligned} \tag{4.5}$$

For any scalar function $\vartheta \in \mathbf{R}$ and any vector function $\varrho \in \mathbf{R}^3$ such that $\varrho|_{\Gamma_h} = 0$, there holds

$$\int_\Omega (\nabla_{\mathcal{A}}\vartheta)\cdot\varrho = \int_{\Gamma_0}\vartheta(\mathcal{N}\cdot\varrho) - \int_\Omega \vartheta\,\mathrm{div}_{\mathcal{A}}\varrho, \tag{4.6}$$

due to the integration by parts and the identities $\mathcal{A}_{i3}\varrho_i = \mathcal{N}\cdot\varrho$ and $\partial_j\mathcal{A}_{ij} = 0$. One deduces from (4.5) that

$$\begin{aligned}&\int_\Omega \nabla_{\mathcal{A}}\partial_t^j q\cdot\partial_t^j u - \int_\Omega \mu\,\mathrm{div}_{\mathcal{A}}\mathbb{S}_{\mathcal{A}}\partial_t^j u\cdot\partial_t^j u \\ &= \int_{\Gamma_0}(\partial_t^j q\mathrm{Id} - \mu\mathbb{S}_{\mathcal{A}}\partial_t^j u)\mathcal{N}\cdot\partial_t^j u - \int_\Omega \mu(\mathrm{div}_{\mathcal{A}}\partial_t^j u)\partial_t^j q + \frac{1}{2}\int_\Omega \mu|\mathbb{S}_{\mathcal{A}}\partial_t^j u|^2 \\ &= \int_{\Gamma_0} g\rho_+\partial_t^j\eta_3\mathcal{N}\cdot\partial_t^j u + \int_{\Gamma_0} F^{3,j}\cdot\partial_t^j u - \int_\Omega F^{2,j}\partial_t^j q + \frac{1}{2}\int_\Omega \mu|\mathbb{S}_{\mathcal{A}}\partial_t^j u|^2.\end{aligned} \tag{4.7}$$

Using the integration by parts and inserting (4.7) into (4.5), we obtain (4.4). □

Lemma 4.2. *The following inequalities hold*

$$\begin{aligned}&\|(F^{1,1},F^{2,1})\|_{L^2(\Omega)} + \|F^{3,1}\|_{L^2(\Gamma_0)} \lesssim_\varepsilon \mathcal{E}^2, \\ &\|(F^{1,2},F^{2,2})\|_{L^2(\Omega)} + \|\partial_t F^{2,2}\|_{L^2(\Omega)} + \|F^{3,2}\|_{L^2(\Gamma_0)} \lesssim_\varepsilon \mathcal{E}(\mathcal{E}+\mathcal{D}).\end{aligned} \tag{4.8}$$

Proof. For $\Sigma = \Omega$ or Γ_0, all quadratic terms $\|X_1X_2\|_{L^2(\Sigma)}$ or cubic ones $\|X_1X_2X_3\|_{L^2(\Sigma)}$ appearing in $F^{j,l}$ with $1 \leqslant j \leqslant 5$ will be bounded by using Sobolev embedding. Precisely, we have

$$\|X_1X_2\|_{L^2(\Sigma)} \lesssim \|X_1\|_{L^\infty(\Sigma)}\|X_2\|_{L^2(\Sigma)} \lesssim \|X_1\|_{H^2(\Sigma)}\|X_2\|_{L^2(\Sigma)}$$

and

$$\|X_1X_2X_3\|_{L^2(\Sigma)} \lesssim \|X_1\|_{L^\infty(\Sigma)}\|X_2\|_{L^\infty(\Sigma)}\|X_3\|_{L^2(\Sigma)} \lesssim \|X_1\|_{H^2(\Sigma)}\|X_2\|_{H^2(\Sigma)}\|X_3\|_{L^2(\Sigma)}.$$

The proof of Lemma 4.2 is followed by that one of [31, Lemma 4.4]. □

Now, we are in position to prove Proposition 4.1.

Proof. Using (4.4), and Sobolev embedding and Cauchy-Schwarz's inequality, we get

$$\begin{aligned}\frac{d}{dt}\Big(\int_\Omega \rho_0|u|^2 + \mathrm{m}^2\int_\Omega |\partial_3\eta|^2 + \int_{\Gamma_0} g\rho_+|\eta_3|^2\Big) + \int_\Omega \mu|\mathbb{S}_{\mathcal{A}}u|^2 \\ = \int_\Omega g\rho_0'\eta_3 u_3 + \int_{\Gamma_0} g\rho_+\eta_3 u\cdot(\mathcal{N}-e_3) \\ \lesssim \|\eta_3\|_{L^2(\Omega)}\|u_3\|_{L^2(\Omega)} + \|\eta_3\|_{L^2(\Gamma_0)}\|u\|_{L^2(\Gamma_0)}\|\mathcal{N}-e_3\|_{H^3(\Omega)}.\end{aligned}$$

Note that thanks to [12, Section 2.1], we have

$$\|\mathcal{N}-e_3\|_{H^s(\Omega)} = \|(\mathcal{A}-\mathrm{Id})e_3\|_{H^s(\Omega)} \lesssim \|\eta\|_{H^{s+1}(\Omega)},$$

yielding

$$\begin{aligned}\frac{d}{dt}\Big(\int_\Omega \rho_0|u|^2 + \mathrm{m}^2\int_\Omega |\partial_3\eta|^2 + \int_{\Gamma_0} g\rho_+|\eta_3|^2\Big) + \int_\Omega \mu|\mathbb{S}_{\mathcal{A}}u|^2 \\ \leqslant C\|\eta_3\|_{L^2(\Omega)}\|u_3\|_{L^2(\Omega)} + C_\varepsilon\mathcal{E}(\mathcal{E}^2+\mathcal{D}^2).\end{aligned}$$

Thanks to the above inequality and Korn's inequality, we integrate in time to obtain $(4.2)_{j=0}$. The proof $(4.2)_{j=1}$ follows the same pattern.

Now, we prove (4.3). Let us use $(4.4)_{j=2}$ and the trace theorem to get

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\Big(\int_\Omega \rho_0|\partial_t^2 u|^2 + \mathrm{m}^2\int_\Omega |\partial_3\partial_t^2\eta|^2 + \int_{\Gamma_0} g\rho_+|\partial_t^2\eta_3|^2\Big) + \frac{1}{2}\int_\Omega \mu|\mathbb{S}_{\mathcal{A}}\partial_t^2 u|^2 \\ &= \int_\Omega g\rho_0' u_3\partial_t u_3 + \int_\Omega F^{1,2}\cdot\partial_t^2 u + \int_\Omega F^{2,2}\cdot\partial_t^2 q - \int_{\Gamma_0} F^{3,2}\cdot\partial_t^2 u - \int_{\Gamma_0} g\rho_+\partial_t^2\eta_3\partial_t^2 u\cdot(\mathcal{N}-e_3) \\ &\lesssim \|u_3\|_{L^2(\Omega)}\|\partial_t u_3\|_{L^2(\Omega)} + (\|F^{1,2}\|_{L^2(\Omega)} + \|F^{3,2}\|_{L^2(\Gamma_0)})\|\partial_t^2 u\|_{H^1(\Omega)} \\ &\quad + \|\partial_t u_3\|_{H^1(\Omega)}\|\partial_t^2 u\|_{H^1(\Omega)}\|\mathcal{N}-e_3\|_{H^3(\Omega)} + \int_\Omega F^{2,2}\partial_t^2 q.\end{aligned}$$

Using (4.8), we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\Big(\int_\Omega \rho_0|\partial_t^2 u|^2 + \mathrm{m}^2\int_\Omega |\partial_3\partial_t^2\eta|^2 + \int_{\Gamma_0} g\rho_+|\partial_t^2\eta_3|^2\Big) + \frac{1}{2}\int_\Omega \mu|\mathbb{S}_{\mathcal{A}}\partial_t^2 u|^2 \\ \leqslant C\|u_3\|_{L^2(\Omega)}\|\partial_t u_3\|_{L^2(\Omega)} + C\int_\Omega F^{2,2}\partial_t^2 q + C_\varepsilon\mathcal{E}(\mathcal{E}^2+\mathcal{D}^2).\end{aligned}$$

Integrating in time, that implies

$$\begin{aligned}&\|\partial_t^2 u(t)\|_{L^2(\Omega)}^2 + \|\partial_t^2\partial_3\eta(t)\|_{L^2(\Omega)}^2 + \|\partial_t^2\eta_3(t)\|_{L^2(\Gamma_0)}^2 + \int_0^t \|\partial_t^2 u(s)\|_{H^1(\Omega)}^2 ds \\ &\leqslant C\big(\|\partial_t^2 u(0)\|_{L^2(\Omega)}^2 + \|\partial_3\partial_t u(0)\|_{L^2(\Omega)}^2 + \|\partial_t u_3(0)\|_{L^2(\Gamma_0)}^2\big) + C\int_0^t \|u_3(s)\|_{L^2(\Omega)}^2 \\ &\quad + C\int_0^t \|\partial_t u_3(s)\|_{L^2(\Omega)}^2 + C_\varepsilon\int_0^t \mathcal{E}(\mathcal{E}^2+\mathcal{D}^2)(s)ds + \int_0^t\int_\Omega (F^{2,2}\partial_t^2 q)(s)ds\end{aligned}$$

Since $\partial_t^2 q$ does not appear in $\mathcal{E}$ and $\mathcal{D}$, we use the integration in time to obtain

$$\int_0^t\int_\Omega (F^{2,2}\partial_t^2 q)(s)ds = \int_\Omega (F^{2,2}\partial_t q)(t) - \int_\Omega (F^{2,2}\partial_t q)(0) - \int_0^t\int_\Omega \partial_t q(s)\partial_t F^{2,2}(s)ds.$$

As a result, we observe

$$\begin{aligned}\int_0^t\int_\Omega (F^{2,2}\partial_t^2 q)(s)ds &\leqslant \|\partial_t q(t)\|_{L^2(\Omega)}\|F^{2,2}(t)\|_{L^2(\Omega)} + \|\partial_t q(0)\|_{L^2(\Omega)}\|F^{2,2}(0)\|_{L^2(\Omega)} \\ &\qquad + \int_0^t \|\partial_t q(s)\|_{L^2(\Omega)}\|\partial_t F^{2,2}(s)\|_{L^2(\Omega)}ds \\ &\lesssim_\varepsilon \mathcal{E}^2(0) + \mathcal{E}^3(t) + \int_0^t \mathcal{E}(\mathcal{E}^2+\mathcal{D}^2)(s)ds.\end{aligned} \tag{4.9}$$

Consequently,

$$\begin{aligned}&\|\partial_t^2 u(t)\|_{L^2(\Omega)}^2 + \|\partial_t^2\partial_3\eta(t)\|_{L^2(\Omega)}^2 + \|\partial_t^2\eta_3(t)\|_{L^2(\Gamma_0)}^2 + \int_0^t \|\partial_t^2 u(s)\|_{H^1(\Omega)}^2 ds \\ &\leqslant C_\varepsilon \mathcal{E}^2(0) + C_\varepsilon\mathcal{E}^3(t) + C\int_0^t \|u_3(s)\|_{L^2(\Omega)}^2 + C\int_0^t \|\partial_t u_3(s)\|_{L^2(\Omega)}^2 + C_\varepsilon \int_0^t \mathcal{E}(\mathcal{E}^2+\mathcal{D}^2)(s)ds.\end{aligned}$$

Combining the above inequality with $(4.2)_{j=1}$, we get (4.3). The proof of Proposition 4.1 is finished. □

4.2. **Horizontal spatial estimates.** In this subsection, we establish the estimates of horizontal spatial derivatives. Let $\beta = (\beta_1,\beta_2) \in \mathbb{N}^2$ and let us apply the horizontal derivative $\partial_h^\beta = \partial_1^{\beta_1}\partial_2^{\beta_2}$ to (2.13), we obtain the following equations since ρ_0 only depends on x_3.

$$\begin{cases}\partial_t\partial_h^\beta\eta = \partial_h^\beta u & \text{in } \Omega,\\ \rho_0\partial_t\partial_h^\beta u + \nabla\partial_h^\beta q - \mu\Delta\partial_h^\beta u - \mathrm{m}^2\partial_3^2\partial_h^\beta\eta - g\rho_0'\partial_h^\beta\eta_3 e_3 = \partial_h^\beta\mathcal{Q}_1 & \text{in } \Omega,\\ \mathrm{div}\partial_h^\beta u = \partial_h^\beta \mathcal{Q}_2 & \text{in } \Omega,\\ (\partial_h^\beta q\mathrm{Id} - \mu\mathbb{S}\partial_h^\beta u)e_3 = \mathrm{m}^2\partial_3\partial_h^\beta\eta + g\rho_+\partial_h^\beta\eta_3 e_3 + \partial_h^\beta\mathcal{Q}_3 & \text{on } \Gamma_0,\\ \partial_h^\beta u = 0 & \text{on } \Gamma_h.\end{cases} \tag{4.10}$$

Lemma 4.3. *The following inequalities hold*

$$\|(\mathcal{Q}_1,\mathcal{Q}_2,\mathcal{Q}_3)\|_{H^4(\Omega)} + \|(\partial_t\mathcal{Q}_1,\partial_t\mathcal{Q}_2,\partial_t\mathcal{Q}_3)\|_{H^2(\Omega)} \lesssim_\varepsilon \mathcal{E}(\mathcal{E}+\mathcal{D}). \tag{4.11}$$

Proposition 4.2. *The following inequalities hold*

$$\begin{aligned}&\|u(t)\|_{0,4,\Omega}^2 + \|\partial_3\eta(t)\|_{0,4,\Omega}^2 + \|\eta_3(t)\|_{0,4,\Gamma_0}^2 + \int_0^t \|u(s)\|_{1,4,\Omega}^2 ds \\ &\leqslant C_\varepsilon\mathcal{E}^2(0) + C\varepsilon^2\int_0^t (\|\eta_3(s)\|_{H^5(\Omega)}^2 + \|u_3(s)\|_{H^4(\Omega)}^2) + C\varepsilon^{-8}\int_0^t \|\eta_3(s)\|_{L^2(\Omega)}^2 \\ &\quad + C_\varepsilon\int_0^t \mathcal{E}(\mathcal{E}^2+\mathcal{D}^2)(s)ds,\end{aligned} \tag{4.12}$$

and

$$\begin{aligned}&\|\partial_t u(t)\|_{0,2,\Omega}^2 + \|\partial_3 u(t)\|_{0,2,\Omega}^2 + \|u_3(t)\|_{0,2,\Gamma_0}^2 + \int_0^t \|\partial_t u(s)\|_{1,2,\Omega}^2 ds \\ &\leqslant C_\varepsilon\mathcal{E}^2(0) + C\varepsilon^2\int_0^t (\|u_3(s)\|_{H^3(\Omega)}^2 + \|\partial_t u_3(s)\|_{H^2(\Omega)}^2)ds + C\varepsilon^{-4}\int_0^t \|u_3(s)\|_{L^2(\Omega)}^2 ds \\ &\quad + C_\varepsilon\int_0^t \mathcal{E}(\mathcal{E}^2+\mathcal{D}^2)(s)ds.\end{aligned} \tag{4.13}$$

Proof. Let us prove (4.12) first. For any $\beta \in \mathbb{N}^2$ with $0 \leqslant |\beta| \leqslant 4$, we compute that

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \rho_0|\partial_h^\beta u|^2 + \int_\Omega (\nabla\partial_h^\beta q - \mu\Delta\partial_h^\beta u)\cdot\partial_h^\beta u - \mathrm{m}^2\int_\Omega \partial_3^2\partial_h^\beta\eta\cdot\partial_h^\beta u$$
$$= \int_\Omega g\rho_0'\partial_h^\beta\eta_3\partial_h^\beta u_3 + \int_\Omega \partial_h^\beta\mathcal{Q}_1\cdot\partial_h^\beta u.$$

Using the integration by parts and the boundary conditions $(4.10)_{4,5}$, we get

$$\frac{1}{2}\frac{d}{dt}\Big(\int_\Omega \rho_0|\partial_h^\beta u|^2 + \mathrm{m}^2\int_\Omega |\partial_3\partial_h^\beta\eta|^2 + \int_{\Gamma_0} g\rho_+|\partial_h^\beta\eta_3|^2\Big) + \frac{1}{2}\int_\Omega \mu|\mathbb{S}\partial_h^\beta u|^2$$
$$= \int_\Omega g\rho_0'\partial_h^\beta\eta_3\partial_h^\beta u_3 - \int_{\Gamma_0}\partial_h^\beta\mathcal{Q}_3\cdot\partial_h^\beta u + \int_\Omega \partial_h^\beta q\partial_h^\beta\mathcal{Q}_2 + \int_\Omega \partial_h^\beta\mathcal{Q}_1\cdot\partial_h^\beta u.$$

Using Cauchy-Schwarz's inequality and (A.4),

$$\int_\Omega g\rho_0'\partial_h^\beta\eta_3\partial_h^\beta u_3 \lesssim \|\eta_3\|_{H^4(\Omega)}\|u_3\|_{H^4(\Omega)} \lesssim \varepsilon^{-8}\|\eta_3\|_{L^2(\Omega)}^2 + \varepsilon^2(\|\eta_3\|_{H^5(\Omega)}^2 + \|u_3\|_{H^4(\Omega)}^2).$$

For the second term, by the trace theorem

$$\int_{\Gamma_0}\partial_h^\beta\mathcal{Q}_3\cdot\partial_h^\beta u \lesssim \|\mathcal{Q}_3\|_{H^{|\beta|-1/2}(\Gamma_0)}\|u_3\|_{H^{|\beta|+1/2}(\Gamma_0)} \lesssim \|\mathcal{Q}_3\|_{H^4(\Omega)}\|u_3\|_{H^5(\Omega)} \lesssim_\varepsilon \mathcal{E}(\mathcal{E}^2 + \mathcal{D}^2)$$

For other terms, using the estimate (4.11) and the trace theorem, we get

$$\int_\Omega \partial_h^\beta q\partial_h^\beta\mathcal{Q}_2 + \int_\Omega \partial_h^\beta\mathcal{Q}_1\cdot\partial_h^\beta u \lesssim \|q\|_{H^4(\Omega)}\|\mathcal{Q}_2\|_{H^4(\Omega)} + \|\mathcal{Q}_1\|_{H^4(\Omega)}\|u\|_{H^4(\Omega)}$$
$$\lesssim_\varepsilon \mathcal{E}(\mathcal{E}^2 + \mathcal{D}^2).$$

Combining the above estimates and integrating in time, we observe

$$\|\partial_h^\beta u(t)\|_{L^2(\Omega)}^2 + \|\partial_3\partial_h^\beta\eta(t)\|_{L^2(\Omega)}^2 + \|\partial_h^\beta\eta_3(t)\|_{L^2(\Gamma_0)}^2 + \int_0^t \|\partial_h^\beta u(s)\|_{H^1(\Omega)}^2 ds$$
$$\leqslant C\big(\|\partial_h^\beta u(0)\|_{L^2(\Omega)}^2 + \|\partial_3\partial_h^\beta\eta(0)\|_{L^2(\Omega)}^2 + \|\partial_h^\beta\eta_3(0)\|_{L^2(\Gamma_0)}^2\big) + C\varepsilon^2\int_0^t (\|\eta_3(s)\|_{H^5(\Omega)}^2 + \|u_3(s)\|_{H^4(\Omega)}^2)$$
$$+ C\varepsilon^{-8}\int_0^t \|\eta_3(s)\|_{L^2(\Omega)}^2 + C_\varepsilon\int_0^t \mathcal{E}(\mathcal{E}^2 + \mathcal{D}^2)(s)ds,$$

that implies (4.12).

Second, we prove (4.13). To do that, we take the derivative in time to $(4.10)_2$ to get that for $\beta \in \mathbb{N}^2, 0 \leqslant |\beta| \leqslant 2$,

$$\begin{cases} \rho_0\partial_t^2\partial_h^\beta u + \nabla\partial_t\partial_h^\beta q - \mu\Delta\partial_h^\beta\partial_t u - \mathrm{m}^2\partial_3^2\partial_h^\beta\partial_t\eta - g\rho_0'\partial_h^\beta\partial_t\eta_3 e_3 = \partial_h^\beta\partial_t\mathcal{Q}_1 & \text{in } \Omega, \\ \mathrm{div}\partial_h^\beta\partial_t u = \partial_h^\beta\partial_t\mathcal{Q}_2 & \text{in } \Omega, \\ (\partial_h^\beta\partial_t q\mathrm{Id} - \mu\mathbb{S}\partial_h^\beta\partial_t u)e_3 = \mathrm{m}^2\partial_3\partial_h^\beta\partial_t\eta + g\rho_+\partial_h^\beta\partial_t\eta_3 e_3 + \partial_h^\beta\partial_t\mathcal{Q}_3 & \text{on } \Gamma_0, \\ \partial_h^\beta\partial_t u = 0 & \text{on } \Gamma_h. \end{cases} \tag{4.14}$$

Hence,

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \rho_0|\partial_h^\beta\partial_t u|^2 + \int_\Omega (\nabla\partial_h^\beta\partial_t q - \mu\Delta\partial_h^\beta\partial_t u)\cdot\partial_h^\beta\partial_t u - \mathrm{m}^2\int_\Omega \partial_3^2\partial_h^\beta\partial_t\eta\cdot\partial_h^\beta\partial_t u$$
$$= \int_\Omega g\rho_0'\partial_h^\beta\partial_t\eta_3\partial_h^\beta\partial_t u_3 + \int_\Omega \partial_h^\beta\partial_t\mathcal{Q}_1\cdot\partial_h^\beta\partial_t u.$$

Using the integration by parts and the boundary conditions (4.14)$_{4,5}$, we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\Big(\int_\Omega \rho_0|\partial_h^\beta\partial_t u|^2 + \mathrm{m}^2\int_\Omega |\partial_3\partial_h^\beta\partial_t\eta|^2 + \int_{\Gamma_0} g\rho_+|\partial_h^\beta u_3|^2\Big) + \frac{1}{2}\int_\Omega \mu|\mathbb{S}\partial_h^\beta\partial_t u|^2 \\ = \int_\Omega g\rho_0'\partial_h^\beta\partial_t\eta_3\partial_h^\beta\partial_t u_3 - \int_{\Gamma_0}\partial_h^\beta\partial_t\mathcal{Q}_3\cdot\partial_h^\beta\partial_t u + \int_\Omega \partial_h^\beta\partial_t q\partial_h^\beta\partial_t\mathcal{Q}_2 + \int_\Omega \partial_h^\beta\partial_t\mathcal{Q}_1\cdot\partial_h^\beta\partial_t u.\end{aligned}$$

Using again Cauchy-Schwarz's inequality and (A.4),

$$\int_\Omega g\rho_0'\partial_h^\beta\partial_t\eta_3\partial_h^\beta\partial_t u_3 \lesssim \|u_3\|_{H^2(\Omega)}\|\partial_t u_3\|_{H^2(\Omega)} \lesssim \varepsilon^{-4}\|u_3\|^2_{L^2(\Omega)} + \varepsilon^2(\|u_3\|^2_{H^3(\Omega)} + \|\partial_t u_3\|^2_{H^2(\Omega)}).$$

For the second integral,

$$\int_{\Gamma_0}\partial_h^\beta\partial_t\mathcal{Q}_3\cdot\partial_h^\beta\partial_t u \lesssim \|\partial_t\mathcal{Q}_3\|_{H^{|\beta|-1/2}(\Gamma_0)}\|\partial_t u\|_{H^{|\beta|+1/2}(\Gamma_0)} \lesssim \|\partial_t\mathcal{Q}_3\|_{H^2(\Omega)}\|\partial_t u\|_{H^3(\Omega)} \lesssim_\varepsilon \mathcal{E}(\mathcal{E}^2+\mathcal{D}^2).$$

For the other terms, using the estimate (4.11) and the trace theorem, we get

$$\begin{aligned}\int_\Omega \partial_h^\beta\partial_t q\partial_h^\beta\partial_t\mathcal{Q}_2 + \int_\Omega \partial_h^\beta\partial_t\mathcal{Q}_1\cdot\partial_h^\beta\partial_t u &\lesssim \|\partial_t q\|_{H^2(\Omega)}\|\partial_t\mathcal{Q}_2\|_{H^2(\Omega)} + \|\partial_t\mathcal{Q}_1\|_{H^2(\Omega)}\|\partial_t u\|_{H^2(\Omega)} \\ &\lesssim_\varepsilon \mathcal{E}(\mathcal{E}^2+\mathcal{D}^2).\end{aligned}$$

Combining the above inequalities and Korn's inequality, we obtain (4.13). Proposition 4.2 is proven. □

Proposition 4.3. *There holds*

$$\|\eta(t)\|^2_{1,4,\Omega} + \int_0^t \|\partial_3\eta(s)\|^2_{0,4,\Omega}ds \leqslant C\begin{pmatrix}\mathcal{E}^2(0) + \|u(t)\|^2_{0,4,\Omega} + \varepsilon\int_0^t\|\eta(s)\|^2_{H^5(\Omega)} \\ +\,\varepsilon^{-4}\int_0^t\|\eta(s)\|^2_{L^2(\Omega)} + \int_0^t\mathcal{E}(\mathcal{E}^2+\mathcal{D}^2)(s)ds\end{pmatrix}. \tag{4.15}$$

Proof. Note that $\eta|_{\Gamma_h} = 0$, using the integration by parts, we have for any $\beta\in\mathcal{N}^2, 0\leqslant|\beta|\leqslant 4$,

$$\begin{aligned}\int_\Omega \rho_0\partial_t\partial_h^\beta u\cdot\partial_h^\beta\eta + \frac{\mu}{2}\int_\Omega \mathbb{S}(\partial_h^\beta u):\mathbb{S}(\partial_h^\beta\eta) + \mathrm{m}^2\int_\Omega|\partial_3\partial_h^\beta\eta|^2 + \int_{\Gamma_0} g\rho_+|\partial_h^\beta\eta_3|^2 - \int_\Omega g\rho_0'|\partial_h^\beta\eta_3|^2 \\ = \int_{\Gamma_0}\partial_h^\beta\eta\cdot\partial_h^\beta\mathcal{Q}_3 + \int_\Omega \partial_h^\beta q\partial_h^\beta\mathcal{Q}_2 + \int_\Omega \partial_h^\beta\mathcal{Q}_1\cdot\partial_h^\beta\eta.\end{aligned}$$

Since $\partial_t\eta = u$, we obtain

$$\begin{aligned}\int_\Omega \rho_0\partial_t(\partial_h^\beta u)\cdot\partial_h^\beta\eta &= \frac{d}{dt}\int_\Omega \rho_0\partial_h^\beta u\cdot\partial_h^\beta\eta - \int_\Omega \rho_0\partial_h^\beta u\cdot\partial_h^\beta\partial_t\eta \\ &= \frac{d}{dt}\int_\Omega \rho_0\partial_h^\beta u\cdot\partial_h^\beta\eta - \int_\Omega \rho_0|\partial_h^\beta u|^2,\end{aligned}$$

and

$$\int_\Omega \mathbb{S}(\partial_h^\beta u):\mathbb{S}(\partial_h^\beta\eta) = \frac{1}{2}\frac{d}{dt}\int_\Omega|\mathbb{S}(\partial_h^\beta\eta)|^2.$$

Those equalities, with similar estimates in the proof of (4.12) imply

$$\begin{aligned}\frac{d}{dt}\Big(\int_\Omega|\mathbb{S}(\partial_h^\beta\eta)|^2 + \int_\Omega \rho_0\partial_h^\beta u\cdot\partial_h^\beta\eta\Big) + \mathrm{m}^2\int_\Omega|\partial_3\partial_h^\beta\eta|^2 + \int_{\Gamma_0} g\rho_+|\partial_h^\beta\eta_3|^2 \\ \leqslant C\int_\Omega \rho_0|\partial_h^\beta u|^2 + C\int_\Omega g\rho_0'|\partial_h^\beta\eta_3|^2 + C_\varepsilon\mathcal{E}(\mathcal{E}^2+\mathcal{D}^2).\end{aligned}$$

Integrating in time and using Korn's inequality, we obtain

$$
\begin{aligned}
&\|\partial_h^\beta \eta(t)\|_{H^1(\Omega)}^2 + \int_0^t \|\partial_3\partial_h^\beta \eta(s)\|_{L^2(\Omega)}^2 + \int_0^t \|\partial_h^\beta \eta_3(s)\|_{L^2(\Gamma_0)}^2 ds \\
&\quad \leqslant C_\varepsilon \mathcal{E}^2(0) - C\int_\Omega \rho_0 \partial_h^\beta u(t)\cdot \partial_h^\beta \eta(t) + C\int_0^t \|\partial_h^\beta \eta(s)\|_{L^2(\Omega)}^2 + C_\varepsilon \int_0^t \mathcal{E}(\mathcal{E}^2+\mathcal{D}^2)(s)ds \\
&\quad \leqslant C_\varepsilon \mathcal{E}^2(0) + \nu\|\partial_h^\beta \eta(t)\|_{L^2(\Omega)}^2 + C\nu^{-1}\|\partial_h^\beta u(t)\|_{L^2(\Omega)}^2 \\
&\qquad + C\int_0^t \|\eta(s)\|_{H^4(\Omega)}^2 + C_\varepsilon \int_0^t \mathcal{E}(\mathcal{E}^2+\mathcal{D}^2)(s)ds.
\end{aligned}
$$

Let ν be sufficiently small, we use (A.4) to obtain further

$$
\begin{aligned}
&\|\partial_h^\beta \eta(t)\|_{H^1(\Omega)}^2 + \int_0^t \|\partial_3\partial_h^\beta \eta(s)\|_{L^2(\Omega)}^2 + \int_0^t \|\partial_h^\beta \eta_3(s)\|_{L^2(\Gamma_0)}^2 ds \\
&\quad \leqslant C_\varepsilon \mathcal{E}^2(0) + C\|\partial_h^\beta u(t)\|_{L^2(\Omega)}^2 + C\int_0^t \|\partial_h^\beta \eta(s)\|_{L^2(\Omega)}^2 + C_\varepsilon \int_0^t \mathcal{E}(\mathcal{E}^2+\mathcal{D}^2)(s)ds \\
&\quad \leqslant C_\varepsilon \mathcal{E}^2(0) + C\|\partial_h^\beta u(t)\|_{L^2(\Omega)}^2 + C\varepsilon\int_0^t \|\eta(s)\|_{H^5(\Omega)}^2 + C\varepsilon^{-4}\int_0^t \|\eta(s)\|_{L^2(\Omega)}^2 \\
&\qquad + C_\varepsilon \int_0^t \mathcal{E}(\mathcal{E}^2+\mathcal{D}^2)(s)ds.
\end{aligned}
$$

that is (4.15). The proof of Proposition 4.3 is complete. □

4.3. **Elliptic estimates.** We use the elliptic estimates (A.1), (A.2) to derive some inequalities.

Proposition 4.4. *We have*

$$
\begin{aligned}
&\|\eta(t)\|_{H^5(\Omega)}^2 + \|u(t)\|_{H^3(\Omega)}^2 + \int_0^t (\|(\eta,u)(s)\|_{H^5(\Omega)}^2 + \|\partial_t u(s)\|_{H^3(\Omega)}^2)ds \\
&\qquad + \int_0^t (\|q(s)\|_{H^4(\Omega)}^2 + \|\partial_t q(s)\|_{H^2(\Omega)}^2)ds \\
&\quad \leqslant C_\varepsilon \mathcal{E}^2(0) + C\int_0^t (\|(\eta,u)(s)\|_{1,4,\Omega}^2 ds + \|\partial_t u(s)\|_{1,2,\Omega}^2)ds \\
&\qquad + C\int_0^t (\|(\eta_3,u)(s)\|_{H^3(\Omega)}^2 + \|\partial_t^2 u(s)\|_{H^1(\Omega)}^2)ds + C_\varepsilon\int_0^t \mathcal{E}(\mathcal{E}^2+\mathcal{D}^2)(s)ds.
\end{aligned} \tag{4.16}
$$

To do that, we need the following lemma.

Lemma 4.4. *For* $1 \leqslant j \leqslant 5$, *we have*

$$\|u\|_{j-\frac{1}{2},5-j,\Gamma_0} \lesssim \|u\|_{1,4,\Omega}, \tag{4.17}$$

and for $1 \leqslant j \leqslant 3$, *we have*

$$\|\partial_t u\|_{j-\frac{1}{2},3-j,\Gamma_0} \lesssim \|\partial_t u\|_{1,2,\Omega}. \tag{4.18}$$

Proof. For any $s \geqslant 0$, there holds

$$\|f\|_{H^{s+1/2}(\Gamma_0)} \lesssim \|f\|_{H^{1/2}(\Gamma_0)} + \sum_{\beta\in\mathbb{N}^2,|\beta|=s} \|\partial_h^\beta f\|_{H^{1/2}(\Gamma_0)}. \tag{4.19}$$

Since $\Gamma_0 = \mathbf{T}^2\times\{0\}$, we exploit the definition of the Sobolev norm on $\mathbf{T}^2$ to have that

$$\|f\|_{H^{s+1/2}(\Gamma_0)}^2 \approx \sum_{n\in(L^{-1}\mathbb{Z})^2} (1+|n|^2)^{s+1/2}|\hat{f}(n)|^2,$$

where $\hat{f}$ is the Fourier series of f. By Cauchy-Schwarz's inequality, one has

$$\|f\|^2_{H^{s+1/2}(\Gamma_0)} \lesssim \sum_{n\in(L^{-1}\mathbb{Z})^2} (1+|n|^2)^{1/2}|\hat{f}(n)|^2 + \sum_{\beta\in\mathbb{N}^2,|\beta|=s}\sum_{n\in(L^{-1}\mathbb{Z})^2} (1+|n|^2)^{1/2}|n^\beta \hat{f}(n)|^2,$$

which immediately yields (4.19).

Using (4.19) and the trace theorem, we have

$$\begin{aligned}\|u\|_{j-\frac12,5-j,\Gamma_0} &\lesssim \sum_{\beta\in\mathbb{N}^2,|\beta|\leqslant 5-j}\Big(\|\partial_h^\beta u\|_{H^{1/2}(\Gamma_0)} + \sum_{\gamma\in\mathbb{N}^2,|\gamma|=j-1}\|\partial_h^{\gamma+\beta} u\|_{H^{1/2}(\Gamma_0)}\Big)\\ &\lesssim \sum_{\beta\in\mathbb{N}^2,|\beta|\leqslant 5-j}\|\partial_h^\beta u\|_{H^{1/2}(\Gamma_0)} + \sum_{\beta\in\mathbb{N}^2,|\beta|\leqslant 4}\|\partial_h^\beta u\|_{H^{1/2}(\Gamma_0)} \lesssim \sum_{\beta\in\mathbb{N}^2,|\beta|\leqslant 4}\|\partial_h^\beta u\|_{H^1(\Omega)},\end{aligned}$$

yielding (4.17). The proof of (4.18) is similar, we omit details. □

Thanks to the above lemmas, we prove Proposition 4.4.

Proof of Proposition 4.4. Let $w = \mu u + \mathrm{m}^2\eta$, rewriting (2.13) as

$$\begin{cases} -\Delta(\mu u + \mathrm{m}^2\eta) + \nabla q = -\mathrm{m}^2(\partial_1^2\eta + \partial_2^2\eta) - \rho_0\partial_t u + g\rho_0'\eta_3 e_3 + \mathcal{Q}_1 & \text{in } \Omega,\\ \mathrm{div} w = \mathcal{Q}_2 + \mathrm{m}^2\mathrm{div}\eta & \text{in } \Omega,\\ w = \mu u & \text{on } \Gamma_0,\\ w = 0 & \text{on } \Gamma_h. \end{cases} \tag{4.20}$$

For $2 \leqslant j \leqslant 5$, applying ∂_h^β with $\beta \in \mathbb{N}^2$ so that $|\beta| \leqslant 5-j$ to the problem (4.20), and then applying the elliptic estimate (A.2), we have

$$\begin{aligned}\|w\|^2_{j,5-j,\Omega} + \|\nabla q\|^2_{j-2,5-j,\Omega} &\lesssim \|\nabla_h^2\eta\|^2_{j-2,5-j,\Omega} + \|(\partial_t u,\eta_3)\|^2_{j-2,5-j,\Omega} + \|\mathcal{Q}_1\|^2_{j-2,5-j,\Omega}\\ &\quad + \|(\mathcal{Q}_2,\mathrm{div}\eta)\|^2_{j-1,5-j,\Omega} + \|u\|^2_{j-\frac12,5-j,\Gamma_0}\\ &\lesssim \|\eta\|^2_{j-1,6-j,\Omega} + \|(\partial_t u,\eta_3)\|^2_{H^3(\Omega)} + \|u\|^2_{1,4,\Omega}\\ &\quad + \|\mathrm{div}\eta\|^2_{H^4(\Omega)} + \|\mathcal{Q}_1\|^2_{H^3(\Omega)} + \|\mathcal{Q}_2\|^2_{H^4(\Omega)}.\end{aligned}$$

It can be seen that

$$\|w\|^2_{j,5-j,\Omega} = \mu^2\|u\|^2_{j,5-j,\Omega} + \mathrm{m}^4\|\eta\|^2_{j,5-j,\Omega} + \mu\mathrm{m}^2\frac{d}{dt}\|\eta\|^2_{j,5-j,\Omega}.$$

Using the advantages of Jacobian identity $\det(I+\nabla\eta) = 1$ as in [31, Lemma 4.6] or [12, Page 18], we obtain the boundedness of $\mathrm{div}\eta$, that is

$$\|\mathrm{div}\eta\|^2_{H^4(\Omega)} \lesssim_\varepsilon \mathcal{E}(\mathcal{E}^2 + \mathcal{D}^2).$$

Together with (4.11), we get further

$$\begin{aligned}&\frac{d}{dt}\|\eta\|^2_{j,5-j,\Omega} + \|(\eta,u)\|^2_{j,5-j,\Omega} + \|\nabla q\|^2_{j-2,5-j,\Omega}\\ &\quad\leqslant C\big(\|\eta\|^2_{j-1,6-j,\Omega} + \|(\partial_t u,\eta_3)\|^2_{H^3(\Omega)} + \|u\|^2_{1,4,\Omega}\big) + C_\varepsilon\mathcal{E}(\mathcal{E}^2+\mathcal{D}^2).\end{aligned}$$

A suitable combination of the above inequalities for $2 \leqslant j \leqslant 5$ implies that

$$\frac{d}{dt}\|\eta\|^2_{H^5(\Omega)} + \|(\eta,u)\|^2_{H^5(\Omega)} + \|q\|^2_{H^4(\Omega)} \leqslant C\big(\|(\eta,u)\|^2_{1,4,\Omega} + \|(\partial_t u,\eta_3)\|^2_{H^3(\Omega)}\big) + C_\varepsilon\mathcal{E}(\mathcal{E}^2+\mathcal{D}^2).$$

Integrating in time, we deduce

$$
\begin{aligned}
&\|\eta(t)\|_{H^5(\Omega)}^2+\int_0^t(\|(\eta,u)(s)\|_{H^5(\Omega)}^2+\|q(s)\|_{H^4(\Omega)}^2)ds\\
&\leqslant C\|\eta(0)\|_{H^5(\Omega)}^2+C\int_0^t(\|(\eta,u)(s)\|_{1,4,\Omega}^2+\|(\partial_t u,\eta_3)(s)\|_{H^3(\Omega)}^2ds)+C_\varepsilon\int_0^t\mathcal{E}(\mathcal{E}^2+\mathcal{D}^2)(s)ds.
\end{aligned}
\tag{4.21}
$$

Next, we have

$$
\begin{cases}
-\Delta(\partial_t w)+\nabla\partial_t q=-\mathrm{m}^2(\partial_1^2u+\partial_2^2u)-\rho_0\partial_t^2u+g\rho_0'u_3e_3+\partial_t\mathcal{Q}_1 & \text{in } \Omega,\\
\mathrm{div}\partial_t w=\partial_t\mathcal{Q}_2+\mathrm{m}^2\mathrm{div}u & \text{in } \Omega,\\
\partial_t w=\mu\partial_t u & \text{on } \Gamma_0,\\
\partial_t w=0 & \text{on } \Gamma_h.
\end{cases}
\tag{4.22}
$$

Mimicking the above arguments to $\partial_t w$ satisfying (4.22) and using (4.18), we observe

$$
\begin{aligned}
&\frac{d}{dt}\|\partial_t\eta\|_{H^3(\Omega)}^2+\|(\partial_t\eta,\partial_t u)\|_{H^3(\Omega)}^2+\|\partial_t q\|_{H^2(\Omega)}^2\\
&\qquad\leqslant C\big(\|\partial_t\eta\|_{1,2}^2+\|(\partial_t^2u,\partial_t\eta_3)\|_{H^1(\Omega)}^2+\|\partial_t u\|_{1,2,\Omega}^2\big)+C_\varepsilon\mathcal{E}(\mathcal{E}^2+\mathcal{D}^2).
\end{aligned}
$$

That implies

$$
\frac{d}{dt}\|\partial_t\eta\|_{H^3(\Omega)}^2+\|(u,\partial_t u)\|_{H^3(\Omega)}^2+\|\partial_t q\|_{H^2(\Omega)}^2\leqslant C\big(\|(\partial_t u,u)\|_{1,2,\Omega}^2+\|\partial_t^2u\|_{H^1(\Omega)}^2\big)+C_\varepsilon\mathcal{E}(\mathcal{E}^2+\mathcal{D}^2).
$$

As a result,

$$
\begin{aligned}
&\|u(t)\|_{H^3(\Omega)}^2+\int_0^t(\|(u,\partial_t u)(s)\|_{H^3(\Omega)}^2+\|\partial_t q(s)\|_{H^2(\Omega)}^2)ds\\
&\leqslant C\|u(0)\|_{H^3(\Omega)}^2+C\int_0^t(\|(\partial_t u,u)(s)\|_{1,2,\Omega}^2+\|\partial_t^2u(s)\|_{H^1(\Omega)}^2)ds+C_\varepsilon\int_0^t\mathcal{E}(\mathcal{E}^2+\mathcal{D}^2)(s)ds.
\end{aligned}
\tag{4.23}
$$

Thanks to (4.21) and (4.23), we get our desired estimate (4.16). □

Proposition 4.5. *There holds*

$$
\|u\|_{H^4(\Omega)}^2+\|q\|_{H^3(\Omega)}^2+\|\partial_t u\|_{H^2(\Omega)}^2+\|\partial_t q\|_{H^1(\Omega)}^2\leqslant C(\|(\partial_t^2u,u)\|_{L^2(\Omega)}^2+\|\eta\|_{H^4(\Omega)}^2)+C_\varepsilon\mathcal{E}^3. \tag{4.24}
$$

Proof. We derive from (2.13) that

$$
\begin{cases}
-\mu\Delta\partial_t u+\nabla\partial_t q=-\rho_0\partial_t^2u+\mathrm{m}^2\partial_3^2u+g\rho_0'u_3e_3+\partial_t\mathcal{Q}_1 & \text{in } \Omega,\\
\mathrm{div}\partial_t u=\partial_t\mathcal{Q}_2 & \text{in } \Omega,\\
(\partial_t q\mathrm{Id}-\mu\mathbb{S}\partial_t u)e_3=\mathrm{m}^2\partial_3 u+g\rho_+u_3e_3+\partial_t\mathcal{Q}_3 & \text{on } \Gamma_0,\\
\partial_t u=0 & \text{on } \Gamma_h.
\end{cases}
\tag{4.25}
$$

Applying the elliptic estimate (A.1) to (4.25), it tells us that

$$
\begin{aligned}
\|\partial_t u\|_{H^2(\Omega)}^2+\|\partial_t q\|_{H^1(\Omega)}^2\lesssim{}&\|\partial_t^2u\|_{L^2(\Omega)}^2+\|(\partial_3^2u,u_3)\|_{L^2(\Omega)}^2+\|(\partial_3^2u,u_3)\|_{H^{1/2}(\Gamma_0)}^2\\
&+\|\partial_t\mathcal{Q}_1\|_{L^2(\Omega)}^2+\|\partial_t\mathcal{Q}_2\|_{H^1(\Omega)}^2+\|\partial_t\mathcal{Q}_3\|_{H^{1/2}(\Gamma_0)}^2.
\end{aligned}
$$

Due to the trace theorem, we have

$$
\begin{aligned}
\|\partial_t u\|_{H^2(\Omega)}^2+\|\partial_t q\|_{H^1(\Omega)}^2\lesssim{}&\|\partial_t^2u\|_{L^2(\Omega)}^2+\|(\partial_3^2u,u_3)\|_{H^1(\Omega)}^2\\
&+\|\partial_t\mathcal{Q}_1\|_{L^2(\Omega)}^2+\|\partial_t\mathcal{Q}_2\|_{H^1(\Omega)}^2+\|\partial_t\mathcal{Q}_3\|_{H^1(\Omega)}^2.
\end{aligned}
$$

Due to (4.11), this yields

$$
\|\partial_t u\|_{H^2(\Omega)}^2+\|\partial_t q\|_{H^1(\Omega)}^2\leqslant C\big(\|\partial_t^2u\|_{L^2(\Omega)}^2+\|u\|_{H^3(\Omega)}^2\big)+C_\varepsilon\mathcal{E}^4. \tag{4.26}
$$

Meanwhile, we rewrite (2.13) as

$$\begin{cases} -\Delta u + \nabla q = -\rho_0 \partial_t u + \mathrm{m}^2 \partial_3^2 \eta + g\rho_0' \eta_3 e_3 + \mathcal{Q}_1 & \text{in } \Omega, \\ \mathrm{div} u = \mathcal{Q}_2 & \text{in } \Omega, \\ (q\mathrm{Id} - \mu \mathbb{S} u) e_3 = \mathrm{m}^2 \partial_3 \eta + g\rho_+ \eta_3 e_3 + \mathcal{Q}_3 & \text{on } \Gamma_0, \\ u = 0 & \text{on } \Gamma_h. \end{cases} \tag{4.27}$$

Owing to (4.11) and by applying the elliptic estimate (A.1) again to (4.27), we observe that

$$\begin{aligned} \|u\|_{H^4(\Omega)}^2 + \|q\|_{H^3(\Omega)}^2 &\lesssim \|\partial_t u\|_{H^2(\Omega)}^2 + \|(\partial_3^2 \eta, \eta_3)\|_{H^2(\Omega)}^2 + \|\mathcal{Q}_1\|_{H^2(\Omega)}^2 \\ &\quad + \|\mathcal{Q}_2\|_{H^3(\Omega)}^2 + \|(\partial_3 \eta, \eta_3)\|_{H^{5/2}(\Gamma)}^2 + \|\mathcal{Q}_3\|_{H^{5/2}(\Gamma_0)}^2 \\ &\leqslant C(\|\partial_t u\|_{H^2(\Omega)}^2 + \|\eta\|_{H^4(\Omega)}^2) + C_\varepsilon \mathcal{E}^4. \end{aligned} \tag{4.28}$$

Combining (4.26) and (4.28) and using (4.11), one has

$$\begin{aligned} &\|u\|_{H^4(\Omega)}^2 + \|q\|_{H^3(\Omega)}^2 + \|\partial_t u\|_{H^2(\Omega)}^2 + \|\partial_t q\|_{H^1(\Omega)}^2 \\ &\quad \leqslant C(\|\partial_t^2 u\|_{L^2(\Omega)}^2 + \|u\|_{H^3(\Omega)}^2 + \|\eta\|_{H^4(\Omega)}^2) + C_\varepsilon \mathcal{E}^4 \\ &\quad \leqslant C(\|\partial_t^2 u\|_{L^2(\Omega)}^2 + \nu \|u\|_{H^4(\Omega)}^2 + \nu^{-3} \|u\|_{L^2(\Omega)}^2 + \|\eta\|_{H^4(\Omega)}^2) + C_\varepsilon \mathcal{E}^4. \end{aligned}$$

Let $\nu > 0$ be sufficiently small, (4.24) thus follows. Proof of Proposition 4.5 is complete. □

4.4. **Proof of Proposition 2.1.** From Propositions 4.1, 4.4, we combine (4.2), (4.3) and (4.16) to get that

$$\begin{aligned} &\|\eta(t)\|_{H^5(\Omega)}^2 + \|u\|_{H^3(\Omega)}^2 + \|(\partial_t u, \partial_t^2 u)(t)\|_{L^2(\Omega)}^2 + \int_0^t (\|(\eta, u)\|_{H^5(\Omega)}^2 + \|\partial_t u(s)\|_{H^3(\Omega)}^2) ds \\ &\qquad + \int_0^t (\|q(s)\|_{H^4(\Omega)}^2 + \|\partial_t q(s)\|_{H^2(\Omega)}^2) ds + \int_0^t \|(u, \partial_t u, \partial_t^2 u)(s)\|_{H^1(\Omega)}^2 ds \\ &\quad \leqslant C_\varepsilon \mathcal{E}^2(0) + C \int_0^t (\|(\eta, u)(s)\|_{1,4,\Omega}^2 + \|\partial_t u(s)\|_{1,2,\Omega}^2) ds + C \int_0^t \|(\eta, u)(s)\|_{H^3(\Omega)}^2 ds \\ &\qquad + C_\varepsilon \int_0^t \mathcal{E}(\mathcal{E}^2 + \mathcal{D}^2)(s) ds. \end{aligned} \tag{4.29}$$

Using (4.2), (4.3) again and using also (4.24),

$$\begin{aligned} &\sum_{j=0}^2 \left(\|\partial_t^j u(t)\|_{H^{4-2j}(\Omega)}^2 + \|\partial_t^j \partial_3 \eta(t)\|_{L^2(\Omega)}^2 \right) + \|q(t)\|_{H^3(\Omega)}^2 + \|\partial_t q(t)\|_{H^1(\Omega)}^2 \\ &\qquad + \int_0^t \|(u, \partial_t u, \partial_t^2 u)(s)\|_{H^1(\Omega)}^2 ds \\ &\leqslant C_\varepsilon \mathcal{E}^2(0) + C\|\eta(t)\|_{H^4(\Omega)}^2 + C_\varepsilon \mathcal{E}^3(t) + C \int_0^t \|(\eta, u)(s)\|_{L^2(\Omega)}^2 + C_\varepsilon \int_0^t \mathcal{E}(\mathcal{E}^2 + \mathcal{D}^2)(s) ds. \end{aligned} \tag{4.30}$$

Thanks to (4.29) and (4.30), we obtain

$$\begin{aligned}
&\|\eta(t)\|_{H^5(\Omega)}^2+\sum_{j=0}^{2}\left(\|\partial_t^j u(t)\|_{H^{4-2j}(\Omega)}^2+\|\partial_t^j\partial_3\eta(t)\|_{L^2(\Omega)}^2\right)+\|q(t)\|_{H^3(\Omega)}^2+\|\partial_t q(t)\|_{H^1(\Omega)}^2\\
&\quad+\int_0^t\left(\|\eta(s)\|_{H^5(\Omega)}^2+\|q(s)\|_{H^4(\Omega)}^2+\|\partial_t q(s)\|_{H^2(\Omega)}^2\right)ds+\sum_{j=0}^{2}\int_0^t\|\partial_t^j u(t)\|_{H^{5-2j}(\Omega)}^2 ds\\
&\lesssim C_\varepsilon\mathcal{E}^2(0)+C_\varepsilon\mathcal{E}^3(t)+C\int_0^t(\|(\eta,u)(s)\|_{1,4,\Omega}^2+\|\partial_t u(s)\|_{1,2,\Omega}^2)ds+C\int_0^t\|(\eta,u)(s)\|_{H^3(\Omega)}^2\\
&\quad+C_\varepsilon\int_0^t\mathcal{E}(\mathcal{E}^2+\mathcal{D}^2)(s)ds.
\end{aligned}\tag{4.31}$$

Meanwhile, thanks to Propositions 4.2, 4.3, a suitable linear combination of (4.12), (4.13) and (4.15) gives us that

$$\begin{aligned}
&\|\eta(t)\|_{1,4,\Omega}^2+\|\partial_3\eta(t)\|_{0,4,\Omega}^2+\|u(t)\|_{0,4,\Omega}^2+\|\partial_t u(t)\|_{0,2,\Omega}^2\\
&\quad+\int_0^t(\|\partial_3\eta(s)\|_{0,4,\Omega}^2+\|u(s)\|_{1,4,\Omega}^2+\|\partial_t u(s)\|_{1,2,\Omega}^2)ds\\
&\leqslant C_\varepsilon\mathcal{E}^2(0)+C\varepsilon^2\int_0^t(\|\eta(s)\|_{H^5(\Omega)}^2+\|u(s)\|_{H^4(\Omega)}^2+\|\partial_t u(s)\|_{H^2(\Omega)}^2)ds\\
&\quad+C\varepsilon^{-8}\int_0^t\|(\eta,u)(s)\|_{L^2(\Omega)}^2+C_\varepsilon\int_0^t\mathcal{E}(\mathcal{E}^2+\mathcal{D}^2)(s)ds
\end{aligned}\tag{4.32}$$

Chaining the above inequalities (4.31) $\times\ \varepsilon$ + (4.32), we conclude our desired estimate (2.29).

5. Nonlinear instability

In this section, to indicate some constants being referred later, we will denote them in particular by $C_i, i\geqslant 1$.

Let $\|U\|_{\mathcal{E}}:=\mathcal{E}(U)$, which is defined as in (2.27). Let

$$F_{\mathsf{N}}(t)=\sum_{j=1}^{\mathsf{N}}|\mathsf{c}_j|e^{\lambda_j t}=\sum_{j=j_m}^{\mathsf{N}}|\mathsf{c}_j|e^{\lambda_j t},$$

and $0<\nu_0\ll 1$ be fixed later (5.26). There exists a unique T^δ such that $\delta F_{\mathsf{N}}(T^\delta)=\nu_0$. Let

$$C_1=\|U^{\mathsf{N}}(0)\|_{\mathcal{E}},\quad C_2=\|(\eta^{\mathsf{N}},u^{\mathsf{N}})(0)\|_{L^2(\Omega)}.\tag{5.1}$$

We define

$$\begin{aligned}
&T^\star:=\sup\Big\{t\in(0,T^{\max})|\|U^{\delta,\mathsf{N}}(t)\|_{\mathcal{E}}\leqslant 2C_1\delta_0\},\\
&T^{\star\star}:=\sup\{t\in(0,T^{\max})|\|(\eta^{\delta,\mathsf{N}},u^{\delta,\mathsf{N}})(t)\|_{L^2(\Omega)}\leqslant 2C_2\delta F_{\mathsf{N}}(t)\Big\}.
\end{aligned}\tag{5.2}$$

Note that

$$\|U^{\delta,\mathsf{N}}(0)\|_{\mathcal{E}}\leqslant\delta\|U^{\mathsf{N}}(0)\|_{\mathcal{E}}+\|U^d(0)\|_{\mathcal{E}}\leqslant C_1\delta+C_{\mathsf{N}}^\star\delta^2<2C_1\delta_0,$$

we then have $T^\star>0$. Similarly, we have $T^{\star\star}>0$.

Next, we derive the bound in time of $\|(\eta^d,u^d)\|_{L^2}$.

Proposition 5.1. *For all $t\leqslant\min\{T^\delta,T^\star,T^{\star\star}\}$, there holds*

$$\|(\eta^d,u^d)(t)\|_{L^2(\Omega)}^2\leqslant C_3\delta^3\Big(\sum_{j=j_m}^{\mathsf{P}}|\mathsf{c}_j|e^{\lambda_j t}+\max(0,\mathsf{N}-\mathsf{P})\max_{\mathsf{P}+1\leqslant j\leqslant\mathsf{N}}|\mathsf{c}_j|e^{\frac{2}{3}\Lambda t}\Big)^3.\tag{5.3}$$

In order to prove Proposition 5.1, we need the following bound in time of $\|U^{\delta,\mathsf{N}}\|_{\mathcal{E}}$.

Lemma 5.1. *For all* $t \leqslant \min\{T^\delta, T^\star, T^{\star\star}\}$, *there holds*

$$\|U^{\delta,\mathsf{N}}(t)\|_{\mathcal{E}} \leqslant C_4 \delta F_{\mathsf{N}}(t). \tag{5.4}$$

Proof. We fix a sufficiently small constant ε such that $C_0\varepsilon \leqslant \frac{\lambda_{\mathsf{N}}}{4}$. Hence, it follows from (2.29) that

$$\begin{aligned}\mathcal{E}^2(U^{\delta,\mathsf{N}}(t)) + \int_0^t \mathcal{D}^2(U^{\delta,\mathsf{N}}(s))ds \leqslant \frac{\lambda_{\mathsf{N}}}{4}\int_0^t \mathcal{E}^2(U^{\delta,\mathsf{N}}(s))ds + C_{\lambda_{\mathsf{N}}}\big(\mathcal{E}^2(U^{\delta,\mathsf{N}}(0)) + \mathcal{E}^3(U^{\delta,\mathsf{N}}(t))\big) \\ + C_{\lambda_{\mathsf{N}}}\Big(\int_0^t \mathcal{E}^2(U^{\delta,\mathsf{N}}(s))ds + \int_0^t \mathcal{E}(\mathcal{E}^2+\mathcal{D}^2)(U^{\delta,\mathsf{N}}(s))ds\Big).\end{aligned} \tag{5.5}$$

Refining δ_0, we get $2C_{\lambda_{\mathsf{N}}}C_1\delta_0 \leqslant \frac{1}{2}$ and $2C_{\lambda_{\mathsf{N}}}C_1\delta_0 \leqslant \frac{\lambda_{\mathsf{N}}}{4}$, one thus has

$$\begin{aligned}&\frac{1}{2}\mathcal{E}^2(U^{\delta,\mathsf{N}}(t)) + \frac{1}{2}\int_0^t \mathcal{D}^2(U^{\delta,\mathsf{N}}(s))ds \\ &\leqslant C_{\lambda_{\mathsf{N}}}\mathcal{E}^2(U^{\delta,\mathsf{N}}(0)) + \Big(\frac{\lambda_{\mathsf{N}}}{4} + 2C_{\lambda_{\mathsf{N}}}C_1\delta\Big)\int_0^t \mathcal{E}^2(U^{\delta,\mathsf{N}}(s))ds + C_{\lambda_{\mathsf{N}}}\int_0^t \|(\eta^{\delta,\mathsf{N}}, u^{\delta,\mathsf{N}})(s)\|^2_{L^2(\Omega)}ds \\ &\leqslant C_{\lambda_{\mathsf{N}}}\mathcal{E}^2(U^{\delta,\mathsf{N}}(0)) + \frac{\lambda_{\mathsf{N}}}{2}\int_0^t \mathcal{E}^2(U^{\delta,\mathsf{N}}(s))ds + C_{\lambda_{\mathsf{N}}}\int_0^t \|(\eta^{\delta,\mathsf{N}}, u^{\delta,\mathsf{N}})(s)\|^2_{L^2(\Omega)}ds.\end{aligned}$$

Hence,

$$\begin{aligned}\|U^{\delta,\mathsf{N}}(t)\|^2_{\mathcal{E}} &\leqslant 2C_{\lambda_{\mathsf{N}}}\|U^{\delta,\mathsf{N}}(0)\|^2_{\mathcal{E}} + \lambda_{\mathsf{N}}\int_0^t \|U^{\delta,\mathsf{N}}(s)\|^2_{\mathcal{E}}ds + 2C_{\lambda_{\mathsf{N}}}\int_0^t \|(\eta^{\delta,\mathsf{N}}, u^{\delta,\mathsf{N}})(s)\|^2_{L^2(\Omega)}ds \\ &\leqslant \lambda_{\mathsf{N}}\int_0^t \|U^{\delta,\mathsf{N}}(s)\|^2_{\mathcal{E}}ds + C_5\delta^2F^2_{\mathsf{N}}(t).\end{aligned}$$

Applying Gronwall's inequality, the resulting inequality tells us that

$$\|U^{\delta,\mathsf{N}}(t)\|^2_{\mathcal{E}} \leqslant C_5\Big(\delta^2F^2_{\mathsf{N}}(t) + \delta^2\int_0^t e^{\lambda_{\mathsf{N}}(t-s)}F^2_{\mathsf{N}}(s)ds\Big). \tag{5.6}$$

Note that $\lambda_{\mathsf{N}} < \lambda_j$ for all $1 \leqslant j \leqslant \mathsf{N}-1$, we have

$$\int_0^t e^{\lambda_{\mathsf{N}}(t-s)}F^2_{\mathsf{N}}(s)ds \leqslant \mathsf{N}^2\sum_{j=j_m}^{\mathsf{N}}\int_0^t e^{\lambda_{\mathsf{N}}(t-s)}|\mathsf{c}_j|^2e^{2\lambda_j s}ds \leqslant \mathsf{N}^2e^{\lambda_{\mathsf{N}}t}\sum_{j=j_m}^{\mathsf{N}}|\mathsf{c}_j|^2\frac{e^{(2\lambda_j-\lambda_{\mathsf{N}})t}}{2\lambda_j-\lambda_{\mathsf{N}}}. \tag{5.7}$$

Substituting (5.7) into (5.6), this yields (5.4). We deduce Proposition 5.1. $\square$

Proof. Differentiating $(2.35)_{2,4}$ with respect to t and then eliminating the terms

$$\begin{cases}\rho_0\partial_t^2u^d - \mu\Delta\partial_tu^d + \nabla\partial_tq^d - \mathrm{m}^2\partial_3^2u^d - g\rho_0'u_3^de_3 = \partial_t\mathcal{Q}_1(U^d) & \text{in } \Omega,\\ \mathrm{div}\partial_tu^d = \partial_t\mathcal{Q}_2(U^d) & \text{in } \Omega,\\ (\partial_tq^d\mathrm{Id} - \mu\mathbb{S}(\partial_tu^d))e_3 = \mathrm{m}^2\partial_3u^d + g\rho_+u_3^de_3 + \partial_t\mathcal{Q}_3(U^d) & \text{on } \Gamma_0,\\ \partial_tu^d = 0 & \text{on } \Gamma_h,\end{cases} \tag{5.8}$$

Multiplying both sides of $(5.8)_1$ by ∂_tu^d, we obtain that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega \rho_0|\partial_tu^d|^2 + \int_\Omega(\nabla\partial_tq^d - \mu\Delta\partial_tu^d)\cdot\partial_tu^d \\ -\mathrm{m}^2\int_\Omega \partial_3^2u^d\cdot\partial_tu^d - \int_\Omega g\rho_0'u_3^d\partial_tu_3^d = \int_\Omega \partial_t\mathcal{Q}_1(U^{\delta,\mathsf{N}})\cdot\partial_tu^d.\end{aligned} \tag{5.9}$$

Using the integration by parts, we deduce from the resulting equality that

$$\frac{1}{2}\frac{d}{dt}\Big(\int_\Omega \rho_0|\partial_t u^d|^2 - \int_\Omega g\rho_0'|u_3^d|^2 + \mathrm{m}^2\int_\Omega |\partial_3 u^d|^2\Big) + \frac{\mu}{2}\int_\Omega |\mathbb{S}\partial_t u^d|^2$$
$$+ \int_{\Gamma_0}\big((\partial_t q\mathrm{Id} - \mu\mathbb{S}\partial_t u^d)e_3 - \mathrm{m}^2\partial_3 u^d\big)\cdot\partial_t u^d = \int_\Omega \partial_t q^d \mathrm{div}\partial_t u^d + \int_\Omega \partial_t\mathcal{Q}_1(U^{\delta,\mathsf{N}})\cdot\partial_t u^d.$$

Substituting $(5.8)_{2,3}$ into (5.9), we obtain

$$\frac{1}{2}\frac{d}{dt}\Big(\int_\Omega \rho_0|\partial_t u^d|^2 - \int_\Omega g\rho_0'|u_3^d|^2 + \int_\Gamma g\rho_+|u_3^d|^2 + \mathrm{m}^2\int_\Omega |\partial_3 u^d|^2\Big) + \frac{\mu}{2}\int_\Omega |\mathbb{S}\partial_t u^d|^2$$
$$= \int_\Omega \partial_t\mathcal{Q}_1(U^{\delta,\mathsf{N}})\cdot\partial_t u^d + \int_\Omega \partial_t q^d\partial_t\mathcal{Q}_2(U^{\delta,\mathsf{N}}) - \int_{\Gamma_0}\partial_t\mathcal{Q}_3(U^{\delta,\mathsf{N}})\cdot\partial_t u^d$$
$$\lesssim (\|\partial_t\mathcal{Q}_1(U^{\delta,\mathsf{N}})\|_{L^2(\Omega)} + \|\partial_t\mathcal{Q}_3(U^{\delta,\mathsf{N}})\|_{H^1(\Omega)})(\|\partial_t u^{\delta,\mathsf{N}}\|_{H^1(\Omega)} + \delta\|\partial_t u^{\mathsf{N}}\|_{H^1(\Omega)})$$
$$+ \|\partial_t\mathcal{Q}_2(U^{\delta,\mathsf{N}})\|_{L^2(\Omega)}(\|\partial_t q^{\delta,\mathsf{N}}\|_{L^2(\Omega)} + \delta\|\partial_t q^{\mathsf{N}}\|_{L^2(\Omega)})$$

In view of (4.11), (5.1) and the definition of U^{N}, we deduce

$$\begin{aligned}&\int_\Omega \rho_0|\partial_t u^d(t)|^2 + \int_0^t\int_\Omega \mu|\mathbb{S}\partial_t u^d(s)|^2 ds\\ &\qquad\leqslant z_1 + \int_\Omega g\rho_0'|u_3^d(t)|^2 - \int_{\Gamma_0} g\rho_+|u_3^d(t)|^2 - \mathrm{m}^2\int_\Omega |\partial_3 u^d|^2 + C_6\delta^3 F_{\mathsf{N}}^3(t),\end{aligned} \tag{5.10}$$

where

$$z_1 = \int_\Omega \rho_0|\partial_t u^d(0)|^2 - \int_\Omega g\rho_0'|u_3^d(0)|^2 + \int_{\Gamma_0} g\rho_+|u_3^d(0)|^2.$$

Note that u^d is not divergence-free, we cannot apply Lemma 3.5 directly. In view of [31, Lemma A.9], we thus decompose u^d as

$$u^d = \underline{w} + \widehat{w}$$

such that $\underline{w}$ is divergence-free and $\widehat{w}$ satisfies that

$$\mathrm{div}\widehat{w} = \mathcal{Q}_2(U^{\delta,\mathsf{N}}) \quad \text{and} \quad \|\widehat{w}\|_{H^s(\Omega)} \lesssim \|\mathcal{Q}_2(U^{\delta,\mathsf{N}})\|_{H^{s-1}(\Omega)} \quad \text{for any } s \geqslant 1.$$

We thus have

$$\begin{aligned}&\int_\Omega g\rho_0'|u_3^d|^2 - \int_{\Gamma_0} g\rho_+|u_3^d|^2 - \mathrm{m}^2\int_\Omega |\partial_3 u^d|^2\\ &= \Big(\int_\Omega g\rho_0'|\underline{w}|^2 - \int_{\Gamma_0} g\rho_+|\underline{w}|^2 - \mathrm{m}^2\int_\Omega |\partial_3\underline{w}|^2\Big) + 2\int_\Omega g\rho_0'\underline{w}\cdot\widehat{w} + \int_\Omega g\rho_0'|\widehat{w}|^2\\ &\quad - 2\int_{\Gamma_0} g\rho_+\underline{w}\cdot\widehat{w} - \int_{\Gamma_0} g\rho_+|\widehat{w}|^2 - 2\mathrm{m}^2\int_\Omega \partial_3\underline{w}\cdot\partial_3\widehat{w} - \mathrm{m}^2\int_\Omega |\partial_3\widehat{w}|^2,\end{aligned} \tag{5.11}$$

and

$$\begin{aligned}\Lambda^2\int_\Omega \rho_0|u_3^d|^2 + \frac{\Lambda}{2}\int_\Omega \mu|\mathbb{S}u_3^d|^2 = {}&\Lambda^2\int_\Omega \rho_0|\underline{w}|^2 + \frac{1}{2}\Lambda\int_\Omega \mu|\mathbb{S}\widehat{w}|^2 + \Lambda^2\Big(2\int_\Omega \rho_0\underline{w}\cdot\widehat{w} + \int_\Omega \rho_0|\widehat{w}|^2\Big)\\ &+ \frac{\Lambda}{2}\Big(2\int_\Omega \mu\mathbb{S}\underline{w} : \mathbb{S}\widehat{w} + \int \mu|\mathbb{S}\widehat{w}|^2\Big).\end{aligned} \tag{5.12}$$

We apply Lemma 3.5 to $\underline{w}$ to obtain

$$\int_\Omega g\rho_0'|\underline{w}|^2 - \int_\Gamma g\rho_+|\underline{w}|^2 - \mathrm{m}^2\int_\Omega |\partial_3 u^d|^2 \leqslant \Lambda^2\int_\Omega \rho_0|\underline{w}|^2 + \frac{1}{2}\Lambda\int_\Omega \mu|\mathbb{S}\underline{w}|^2. \tag{5.13}$$

By the trace theorem and (4.11)-(5.4), we have

$$\begin{aligned}
&-\Lambda^2\int_\Omega \rho_0 \underline{w}\cdot\widehat{w} - \frac{1}{2}\Lambda\int_\Omega \mu\mathbb{S}\underline{w} : \mathbb{S}\widehat{w} + \int_\Omega g\rho_0'\underline{w}\cdot\widehat{w} - \int_{\Gamma_0} g\rho_+\underline{w}\cdot\widehat{w} - \mathrm{m}^2\int_\Omega \partial_3\underline{w}\cdot\partial_3\widehat{w}\\
&\quad\lesssim \|u^d\|_{H^1(\Omega)}\|\widehat{w}\|_{H^1(\Omega)} + \|\widehat{w}\|^2_{H^1(\Omega)}\\
&\quad\lesssim (\|u^{\delta,\mathsf{N}}\|_{H^1(\Omega)} + \delta\|u^{\mathsf{N}}\|_{H^1(\Omega)})\|\mathcal{Q}_2(U^{\delta,\mathsf{N}})\|_{L^2(\Omega)} + \|\mathcal{Q}_2(U^{\delta,\mathsf{N}})\|^2_{L^2(\Omega)}\\
&\quad\lesssim \delta^3 F_{\mathsf{N}}^3,
\end{aligned} \tag{5.14}$$

and

$$\begin{aligned}
&-\Lambda^2\int_\Omega \rho_0|\widehat{w}|^2 - \frac{1}{2}\Lambda\int_\Omega \mu|\mathbb{S}\widehat{w}|^2 + \int_\Omega g\rho_0'|\widehat{w}|^2 + \int_{\Gamma_0} g\rho_+|\widehat{w}|^2 + \mathrm{m}^2\int_\Omega |\partial_3\widehat{w}|^2\\
&\quad\lesssim \|\widehat{w}\|^2_{H^1(\Omega)} \lesssim \|\mathcal{Q}_2(U^{\delta,\mathsf{N}})\|^2_{L^2(\Omega)} \lesssim \delta^4 F_{\mathsf{N}}^4,
\end{aligned} \tag{5.15}$$

Plugging (5.13), (5.14) and (5.15) into (5.11), (5.12) we arrive at

$$\int_\Omega g\rho_0'|u_3^d|^2 - \int_{\Gamma_0} g\rho_+|u_3^d|^2 - \mathrm{m}^2\int_\Omega |\partial_3 u^d|^2 \leqslant \Lambda^2\int_\Omega \rho_0|u_3^d|^2 + \frac{1}{2}\Lambda\int_\Omega \mu|\mathbb{S}u_3^d|^2 + C_7\delta^3 F_{\mathsf{N}}^3. \tag{5.16}$$

It follows from (5.16) and (5.10) that

$$\begin{aligned}
\int_\Omega \rho_0|\partial_t u^d(t)|^2 + \int_0^t\int_\Omega \mu|\mathbb{S}\partial_t u^d(s)|^2 ds \leqslant z_1 + \Lambda^2\int_\Omega \rho_0|u^d(t)|^2 + \frac{1}{2}\Lambda\int_\Omega \mu|\mathbb{S}u^d(t)|^2\\
+ (C_6+C_7)\delta^3 F_{\mathsf{N}}^3(t).
\end{aligned} \tag{5.17}$$

Using Cauchy-Schwarz's inequality, we have that

$$\begin{aligned}
\int_\Omega \mu|\mathbb{S}u^d(t)|^2 &= \int_\Omega \mu|\mathbb{S}u^d(0)|^2 + 2\int_0^t\int_\Omega \mu\mathbb{S}u^d(s) : \mathbb{S}\partial_t u^d(s)ds\\
&\leqslant \int_\Omega \mu|\mathbb{S}u^d(0)|^2 + \frac{1}{\Lambda}\int_0^t\int_\Omega \mu|\mathbb{S}\partial_t u^d(s)|^2 ds + \Lambda\int_0^t\int_\Omega \mu|\mathbb{S}u^d(s)|^2 ds
\end{aligned} \tag{5.18}$$

and that

$$\frac{d}{dt}\int_\Omega \rho_0|u^d|^2 \leqslant \frac{1}{\Lambda}\int_\Omega \rho_0|\partial_t u^d|^2 + \Lambda\int_\Omega \rho_0|u^d|^2. \tag{5.19}$$

Three above inequalities (5.17), (5.18) and (5.19) imply that

$$\begin{aligned}
\frac{d}{dt}\int_\Omega \rho_0|u^d(t)|^2 + \frac{1}{2}\int_\Omega \mu|\mathbb{S}u^d(t)|^2 \leqslant \frac{z_1}{\Lambda} + \int_\Omega \mu|\mathbb{S}u^d(0)|^2 + 2\Lambda\int_\Omega \rho_0|u^d(t)|^2\\
+ \Lambda\int_0^t\int_\Omega \mu|\mathbb{S}u^d(s)|^2 ds + (C_6+C_7)\delta^3 F_{\mathsf{N}}^3(t),
\end{aligned} \tag{5.20}$$

It follows from $U^d(0) = \delta^2 U_\star^{\delta,\mathsf{N}}$ that $z_1 \lesssim \delta^4$, this yields

$$\frac{z_1}{\Lambda} + \int_\Omega \mu|\mathbb{S}u^d(0)|^2 \lesssim \delta^4.$$

Hence, the inequality (5.20) implies

$$\frac{d}{dt}\int_\Omega \rho_0|u^d(t)|^2 + \frac{1}{2}\int_\Omega \mu|\mathbb{S}u^d(t)|^2 \leqslant 2\Lambda\int_\Omega \rho_0|u^d(t)|^2 + \Lambda\int_0^t\int_\Omega \mu|\mathbb{S}u^d(s)|^2 ds + C_8\delta^3 F_{\mathsf{N}}^3(t). \tag{5.21}$$

In view of Gronwall's inequality, we obtain from (5.21) that

$$\begin{aligned}\int_\Omega \rho_0|u^d(t)|^2+\frac{1}{2}\int_0^t\int_\Omega \mu|\mathbb{S}u^d(s)|^2ds &\leqslant C_8\delta^3\int_0^t e^{2\Lambda(t-s)}F_{\mathsf{N}}^3(s)ds\\ &\leqslant C_9\delta^3\int_0^t e^{2\Lambda(t-s)}F_{\mathsf{N}}(3s)ds.\end{aligned}\tag{5.22}$$

Due to (2.30), we obtain for $1\leqslant j\leqslant \mathsf{P}$,

$$\int_0^t e^{(3\lambda_j-2\Lambda)s}ds=\frac{1}{3\lambda_j-2\Lambda}(e^{(3\lambda_j-2\Lambda)t}-1)\leqslant\frac{1}{3\lambda_j-2\Lambda}e^{(3\lambda_j-2\Lambda)t}$$

and for $j\geqslant \mathsf{P}+1$,

$$\int_0^t e^{(3\lambda_j-2\Lambda)s}ds=\frac{1}{3\lambda_j-2\Lambda}(e^{(3\lambda_j-2\Lambda)t}-1)\leqslant\frac{1}{2\Lambda-3\lambda_j}.$$

Substituting the two above estimates into (5.22), we observe that

$$\|u^d(t)\|^2_{L^2(\Omega)}+\int_0^t\|\nabla u^d(s)\|^2_{L^2(\Omega)}ds\leqslant C_9\delta^3\Big(\sum_{j=j_m}^{\mathsf{P}}\frac{|\mathsf{c}_j|}{3\lambda_j-2\Lambda}e^{3\lambda_j t}+\sum_{j=\mathsf{P}+1}^{\mathsf{N}}\frac{|\mathsf{c}_j|}{2\Lambda-3\lambda_j}e^{2\Lambda t}\Big).\tag{5.23}$$

To estimate of $\|\eta^d(t)\|_{L^2(\Omega)}$, note that

$$\frac{d}{dt}\|\eta^d(t)\|_{L^2(\Omega)}\leqslant\|\partial_t\eta^d(t)\|_{L^2(\Omega)}=\|u^d\|_{L^2(\Omega)}.$$

Hence, we have that $\|\eta^d(t)\|^2_{L^2(\Omega)}$ is bounded above like (5.23). Proposition 5.1 is proven. □

Now, we are able to conclude the nonlinear RT instability.

Proof of Theorem 2.3. Since $j_m=\min\{j: 1\leqslant j\leqslant\mathsf{P},\mathsf{c}_j\neq 0\}$, we have

$$\begin{aligned}\|u^{\mathsf{N}}(t)\|^2_{L^2(\Omega)}&=\sum_{i=j_m}^{\mathsf{N}}\mathsf{c}_i^2e^{2\lambda_i t}\|u_i\|^2_{L^2(\Omega)}+2\sum_{j_m\leqslant i<j\leqslant\mathsf{N}}\mathsf{c}_i\mathsf{c}_je^{(\lambda_i+\lambda_j)t}\int_\Omega u_i\cdot u_j\\ &\geqslant\sum_{j=j_m}^{\mathsf{N}}\mathsf{c}_j^2e^{2\lambda_j t}\|u_j\|^2_{L^2(\Omega)}+2\sum_{j_m+1\leqslant i<j\leqslant\mathsf{N}}\mathsf{c}_i\mathsf{c}_je^{(\lambda_i+\lambda_j)t}\int_\Omega u_i\cdot u_j\\ &\quad-|\mathsf{c}_{j_m}|\|u_{j_m}\|_{L^2(\Omega)}\Big(\sum_{j=j_m+1}^{\mathsf{N}}|\mathsf{c}_j|\|u_j\|_{L^2(\Omega)}\Big)e^{(\lambda_{j_m}+\lambda_{j_m+1})t}.\end{aligned}$$

By Cauchy-Schwarz's inequality, we obtain

$$\begin{aligned}2\sum_{j_m+1\leqslant i<j\leqslant\mathsf{N}}\mathsf{c}_i\mathsf{c}_je^{(\lambda_i+\lambda_j)t}\int_\Omega u_i\cdot u_j&\geqslant-2\sum_{j_m\leqslant i<j\leqslant\mathsf{N}}|\mathsf{c}_i||\mathsf{c}_j|e^{(\lambda_{j_m+1}+\lambda_{j_m+2})t}\|u_i\|_{L^2(\Omega)}\|u_j\|_{L^2(\Omega)}\\ &\geqslant-e^{(\lambda_{j_m+1}+\lambda_{j_m+2})t}\Big(\sum_{j=j_m+1}^{\mathsf{N}}|\mathsf{c}_j|\|u_j\|_{L^2(\Omega)}\Big)^2.\end{aligned}$$

This implies

$$\begin{aligned}\|u^{\mathsf{N}}(t)\|^2_{L^2(\Omega)}\geqslant&\sum_{j=j_m}^{\mathsf{N}}\mathsf{c}_j^2e^{2\lambda_j t}\|u_j\|^2_{L^2(\Omega)}-e^{(\lambda_{j_m+1}+\lambda_{j_m+2})t}\Big(\sum_{j=j_m+1}^{\mathsf{N}}|\mathsf{c}_j|\|u_j\|_{L^2(\Omega)}\Big)^2\\ &-|\mathsf{c}_{j_m}|e^{(\lambda_{j_m}+\lambda_{j_m+1})t}\|u_{j_m}\|_{L^2(\Omega)}\Big(\sum_{j=j_m+1}^{\mathsf{N}}|\mathsf{c}_j|\|u_j\|_{L^2(\Omega)}\Big).\end{aligned}$$

Due to the assumption (2.33), we deduce that

$$\begin{aligned}\|u^{\mathsf{N}}(t)\|^2_{L^2(\Omega)} \geqslant \sum_{j=j_m}^{\mathsf{N}} \mathsf{c}_j^2 e^{2\lambda_j t}\|u_j\|^2_{L^2(\Omega)} - \mathsf{c}_{j_m}^2 e^{(\lambda_{j_m}+\lambda_{j_m+1})t}\|u_{j_m}\|^2_{L^2(\Omega)} \\ - \mathsf{c}_{j_m}^2 e^{(\lambda_{j_m+1}+\lambda_{j_m+2})t}\|u_{j_m}\|^2_{L^2(\Omega)}.\end{aligned}$$

This yields

$$\begin{aligned}\|u^{\mathsf{N}}(t)\|^2_{L^2(\Omega)} \geqslant \mathsf{c}_{j_m}^2 \Big(e^{2\lambda_{j_m} t} - \frac{1}{2} e^{(\lambda_{j_m}+\lambda_{j_m+1})t} - \frac{1}{4} e^{(\lambda_{j_m+1}+\lambda_{j_m+2})t}\Big)\|u_{j_m}\|^2_{L^2(\Omega)} \\ + \sum_{j=j_m+1}^{\mathsf{N}} \mathsf{c}_j^2 e^{2\lambda_j t}\|u_j\|^2_{L^2(\Omega)}.\end{aligned}$$

Notice that for all $t \geqslant 0$,

$$e^{2\lambda_{j_m} t} - \frac{1}{2} e^{(\lambda_{j_m}+\lambda_{j_m+1})t} - \frac{1}{4} e^{(\lambda_{j_m+1}+\lambda_{j_m+2})t} \geqslant \frac{1}{4} e^{2\lambda_{j_m} t}.$$

Hence, for all $t \leqslant \min\{T^\delta, T^\star, T^{\star\star}\}$, we have

$$\|u^{\mathsf{N}}(t)\|_{L^2(\Omega)} \geqslant C_{10} F_{\mathsf{N}}(t). \tag{5.24}$$

Let $\tilde{\mathsf{c}}(\mathsf{N}) = \max_{\mathsf{P}+1\leqslant j\leqslant \mathsf{N}} \frac{|\mathsf{c}_j|}{|\mathsf{c}_{j_m}|} \geqslant 0$. Now, we show that

$$T^\delta \leqslant \min\{T^\star, T^{\star\star}\} \tag{5.25}$$

by choosing

$$\nu_0 < \min\Big(\frac{2C_1\delta_0}{C_4}, \frac{C_2^2}{C_3(1+\mathsf{N}\tilde{\mathsf{c}}(\mathsf{N}))^3}, \frac{C_{10}^2}{4C_3(1+\mathsf{N}\tilde{\mathsf{c}}(\mathsf{N}))^2}\Big). \tag{5.26}$$

Indeed, if $T^\star < T^\delta$, we have from (5.4) that

$$\|U^{\delta,\mathsf{N}}(T^\star)\|_{\mathcal{E}_f} \leqslant C_4\delta F_{\mathsf{N}}(T^\star) < C_4\delta F_{\mathsf{N}}(T^\delta) = C_4\nu_0 < 2C_1\delta_0,$$

which contradicts the definition of $T^\star$ in (5.2). If $T^{\star\star} < T^\delta$, we obtain from the definition of C_2 (5.1) and the inequality (5.3) that

$$\begin{aligned}&\|(\eta^{\delta,\mathsf{N}}, u^{\delta,\mathsf{N}})(T^\delta)\|_{L^2(\Omega)} \\ &\leqslant \|(\eta^d, u^d)(T^\delta)\|_{L^2(\Omega)} + \delta(\|(\eta^{\mathsf{N}}, u^{\mathsf{N}})(T^\delta)\|_{L^2(\Omega)} \\ &\leqslant \sqrt{C_3}\delta^{\frac{3}{2}}\Big(\sum_{j=j_m}^{\mathsf{P}} |\mathsf{c}_j| e^{\lambda_j T^\delta} + (\mathsf{N}-\mathsf{P})\Big(\max_{\mathsf{P}+1\leqslant j\leqslant \mathsf{N}} |\mathsf{c}_j|\Big) e^{2\Lambda T^\delta/3}\Big)^{3/2} + C_2\delta F_{\mathsf{N}}(T^\delta).\end{aligned} \tag{5.27}$$

Notice from (2.30) that for $\mathsf{P}+1 \leqslant j \leqslant \mathsf{N}$,

$$|\mathsf{c}_j|\delta e^{\frac{2}{3}\Lambda T^\delta} < \frac{|\mathsf{c}_j|}{|\mathsf{c}_{j_m}|}(\delta|\mathsf{c}_{j_m}| e^{\lambda_{j_m} T^\delta}) < \frac{|\mathsf{c}_j|}{|\mathsf{c}_{j_m}|}\delta F_{\mathsf{N}}(T^\delta) = \frac{|\mathsf{c}_j|}{|\mathsf{c}_{j_m}|}\nu_0.$$

Then, it follows from (5.27) that

$$\begin{aligned}\|(\eta^{\delta,\mathsf{N}}, u^{\delta,\mathsf{N}})(T^\delta)\|_{L^2(\Omega)} &\leqslant C_2\delta F_{\mathsf{N}}(T^\delta) + \sqrt{C_3}\delta^{3/2}(1+\mathsf{N}\tilde{\mathsf{c}}(\mathsf{N}))^{3/2} F_{\mathsf{N}}^{3/2}(T^\delta) \\ &\leqslant C_2\nu_0 + \sqrt{C_3}(1+\mathsf{N}\tilde{\mathsf{c}}(\mathsf{N}))^{3/2}\nu_0^{3/2}.\end{aligned}$$

Using (5.26) again, we deduce

$$\|(\eta^{\delta,\mathsf{N}}, u^{\delta,\mathsf{N}})(T^\delta)\|_{L^2(\Omega)} < 2C_2\nu_0 = 2C_2\delta F_{\mathsf{N}}(T^\delta),$$

which also contradicts the definition of $T^{\star\star}$ in (5.2). So, (5.25) holds.

Once we have (5.25), it follows from (5.3) and (5.24) that

$$\begin{aligned}&\|u^{\delta,\mathsf{N}}(T^\delta)\|_{L^2(\Omega)}\\&\geqslant \delta\|u^{\mathsf{N}}(T^\delta)\|_{L^2(\Omega)}-\|u^d(T^\delta)\|_{L^2(\Omega)}\\&\geqslant C_{10}\delta F_{\mathsf{N}}(T^\delta)-\sqrt{C_3}\delta^{3/2}\Big(\sum_{j=j_m}^{\mathsf{P}}|\mathsf{c}_j|e^{\lambda_j T^\delta}+(\mathsf{N}-\mathsf{P})\Big(\max_{\mathsf{P}+1\leqslant j\leqslant \mathsf{N}}|\mathsf{c}_j|\Big)e^{2\Lambda T^\delta/3}\Big)^{3/2}.\end{aligned}$$

Thanks to (5.26) again, we conclude that

$$\|u^{\delta,\mathsf{N}}(T^\delta)\|_{L^2(\Omega)}\geqslant C_{10}\nu_0-\sqrt{C_3}(1+\mathsf{N}\tilde{\mathsf{c}}(\mathsf{N}))^{3/2}\nu_0^{3/2}\geqslant\frac{1}{2}C_{10}\nu_0>0.$$

Theorem 2.3 follows by taking ν_0 satisfying (5.26) and $m_0=\frac{1}{2}C_{10}$. We complete the proof. □

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APPENDIX A. SOME USEFUL ESTIMATES

Elliptic estimates. Let $r \geqslant 2$ and $\phi \in H^{r-2}(\Omega), \psi \in H^{r-1}(\Omega)$ and $\alpha \in H^{r-3/2}(\Gamma)$. There exist unique $u \in H^r(\Omega)$ and $q \in H^{r-1}(\Omega)$ solving

$$\begin{cases} -\Delta u + \nabla q = \phi & \text{in } \Omega, \\ \operatorname{div} u = \psi & \text{in } \Omega, \\ (q\mathrm{Id} - \mu \mathbb{S}u)e_3 = \alpha & \text{on } \Gamma_0, \\ u = 0 & \text{on } \Gamma_h. \end{cases}$$

Moreover, we have

$$\|u\|^2_{H^r(\Omega)} + \|q\|^2_{H^{r-1}(\Omega)} \lesssim \|\phi\|^2_{H^{r-2}(\Omega)} + \|\psi\|^2_{H^{r-1}(\Omega)} + \|\alpha\|^2_{H^{r-3/2}(\Gamma)}. \tag{A.1}$$

thanks to [7, Lemma A.15] for example.

We also recall the classical regularity theory for the Stokes problem with Dirichlet boundary conditions (see [30, Theorem 2.4] after using the domain expansion technique). Let $r \geqslant 2$ and $f \in H^{r-2}(\Omega), g \in H^{r-1}(\Omega)$ and $h \in H^{r-1/2}(\Gamma_0)$ such that

$$\int_\Omega g = \int_{\Gamma_0} h \cdot \nu, \text{ where } \nu \text{ is the outward unit normal vector to the boundary.}$$

There exist uniquely $u \in H^r(\Omega)$ and $q \in H^{r-1}(\Omega)$ solving

$$\begin{cases} -\Delta u + \nabla q = f & \text{in } \Omega, \\ \operatorname{div} u = g & \text{in } \Omega, \\ u = h & \text{on } \Gamma_0, \\ u = 0 & \text{on } \Gamma_h. \end{cases}$$

There also holds

$$\|u\|^2_{H^r(\Omega)} + \|q\|^2_{H^{r-1}(\Omega)} \lesssim \|f\|^2_{H^{r-2}(\Omega)} + \|g\|^2_{H^{r-1}(\Omega)} + \|h\|^2_{H^{r-1/2}(\Gamma)}. \tag{A.2}$$

Korn's inequality. The following Korn's inequality is proven in [18, Theorem 5.12],

$$\|\nabla u\|_{L^2(\Omega)}^2 \lesssim \|\mathbb{S}u\|_{L^2(\Omega)}^2. \tag{A.3}$$

Interpolation inequality. It can be found in [1, Chapter 5] that

$$\|u\|_{H^j(\Omega)} \lesssim \|u\|_{L^2(\Omega)}^{1/(j+1)} \|u\|_{H^{j+1}(\Omega)}^{j/(j+1)}.$$

That implies for $\varepsilon > 0$, there is a universal constant $C(j)$ such that

$$\|u\|_{H^j(\Omega)} \leqslant \varepsilon \|u\|_{H^{j+1}(\Omega)} + C(j)\varepsilon^{-j} \|u\|_{L^2(\Omega)}. \tag{A.4}$$

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