Hermitian algebraic K–theory, Wagoner complex, and the root system D

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Abstract

This manuscript is a based on some recent results of ongoing project which is devoted to investigation of the role of root systems, Weyl and Coxeter groups in algebraic K-theory.

For the root system D we construct an analog of the Wagoner complex used in his proof of the equivalence of K_*^Q and K_*^{BN} (linear) algebraic K-theories. We prove that the corresponding K-theory KU_*^D for the even orthogonal group is naturally isomorphic to KU_*^{BN}-theory constructed by Yu.P. Solovyov and A.I. Nemytov. Also some open problems are raised.

Introduction

Let A be an associative ring. After pioneering works on the algebraic $K_n(A)$ groups for $n = 0, 1, 2$ several definitions of higher algebraic K-groups were proposed. The question of comparing the definitions of the higher K-groups was very natural. Most attention was paid to the sequence of functors and natural transformations described for example in [6]:

$$K_*^Q \to K_*^V \to K_*^S \to K_i^{K-V},$$

The first natural transformation lately was decomposed in [5, 7] into composition of two natural transformarions

$$K_*^Q(A) \xrightarrow{i} K_*^{BN}(A) \to K_*^V(A).$$

Some transformations were proved to be equivalences with less difficulties than other (for K_i^{K-V} one should assume that the argument ring A is left regular).

One of the most interesting case was the equivalence of the Quillen K-theory and the Volodin K-theory which was proved in [3]. Very remarkable proof of this equivalence was found later, see [4].

The groups K_*^{BN} were introduced by Wagoner in [7] and they are a version of the Volodin K–theory K_*^V. The proof in [5] uses the combinatoric of the root system A and the combinatoric of the corresponding partition of $\mathbb{R}^n$ into facettes.

In 1980 Yu.P.Solovyov and A.I.Nemytov for the rings with involution had established natural equivalence of KU_*^Q (Hermitian analog of the Quillen K-theory) and the Hermitian analog of K_*^{BN}-theory. Their construction and the proof were based on the combinatoric of the root system C. In this paper we consider an analog KU_*^D of the K_*^{BN}-theory which is constructed on the root system D and show that for the even orthogonal group this K–theory is equivalent to KU_*^Q.

1 Basic definitions

Let A be an associative ring with 1 equipped with an involution $a \mapsto a^*$ satisfying conditions: (1) $1^* = 1$; (2) $a^{**} = a$; (3) $(a+b)^* = a^* + b^*$; (4) $(ab)^* = b^* a^*$. Let us also fix a central element ε such that $\varepsilon^*\varepsilon = 1 = \varepsilon\varepsilon^*$. Fix an additive subgroup $\Lambda \subset A$ such that

(1) $a\Lambda a^* \subset \Lambda$ for all $a \in A$;

(2) $\Lambda_{min} = \{a - \varepsilon a^* \,:\, a \in A\} \subset \Lambda \subset \Lambda_{max} = \{a \in A \,:\, a = -\varepsilon a^*\}$.

Denote by Λ_{2n} the additive subgroup of $M_{2n}(A)$ consisting of matrices (x_{ij}) with elements satisfying the relations $x_{ij} = -\varepsilon x_{ji}^*$ и $x_{ii} \in \Lambda$.

The set of matrices

$$U_{2n}(A) = U_{2n}(A, \varepsilon, \Lambda) = \\ = \{X \in GL_{2n}(A) \,:\, X^* \begin{pmatrix} 0 & E \\ 0 & 0 \end{pmatrix} X = \begin{pmatrix} 0 & E \\ 0 & 0 \end{pmatrix} \operatorname{mod} \Lambda_{2n}\},$$

with matrix multiplication is call a *unitary group*. It depends on the choice of ε and Λ but for simplicity we shall denote it by $U_{2n}(A)$. Also it is often denoted by ${}_\varepsilon GQ_{2n}(A, \Lambda)$. This definition is due to Bak [**?**]. For particular choices of the parameters ε and Λ one can obtain classical groups like the general linear group, the symplectic group and the even orthogonal group etc.

Passing to the limit with respect to the standard embedding $U_{2n}(A) \to U_{2(n+1)}(A)$ one obtains the group $U(A)$. Define the elementary subgroup to be the subgroup generated by elementary matricies that is the matrices of the form

$$s_{ij}(a) = \begin{pmatrix} 1 + aE_{ij} & 0 \\ 0 & 1 - a^* E_{ji} \end{pmatrix},$$

$$r_{ij}(a) = \begin{pmatrix} 1 & aE_{ij} - \varepsilon^* a^* E_{ji} \\ 0 & 1 \end{pmatrix},$$

$$t_{ij}(a) = \begin{pmatrix} 1 & 0 \\ aE_{ij} - \varepsilon a^* E_{ji} & 1 \end{pmatrix},$$

$$p_i(b) = \begin{pmatrix} 1 & bE_{ii} \\ 0 & 1 \end{pmatrix},$$

$$q_i(c) = \begin{pmatrix} 1 & 0 \\ cE_{ii} & 1 \end{pmatrix},$$

where $a \in A$, $b^*, c \in \Lambda$. It is known (see for example [2]) that $EU(A) = [U(A), U(A)] = [EU(A), EU(A)]$, that is $EU(A)$ is a perfect subgroup and it coincides with the commutant of $U(A)$.

Applying the plus-construction one obtains the definition of the Quillen hermitian K–theory: $KU_j^Q(A) = \pi_j(BU(A)^+)$.

Now let us remind the definition of $KU_*^{BN}(A)$ (see. [1]). Consider the hyperplanes in $\mathbb{R}^n$ given by the equations $e_i \pm e_j = 0$, $1 \le i < j \le n$, and $e_j = 0$, where e_i is the dual basis. Let us call by the facette of codimention j a component in the complement of the union of all $(j+1)$–fold intersection of the hyperplanes in the union of all j-fold intersections. Define the ordering of the facettes: $F < G$ iff $F \subseteq \overline{G}$.

Define $\mathcal{P}_C^n$ to be the simplicial complex with k–simplices of the form $F_0 < F_1 < \ldots < F_k$. The inclusion $\mathcal{P}_C^n \to \mathcal{P}_C^{n+1}$ is induced by the repetition of the last coordinate of a point in $\mathbb{R}^n$. Passing to the limit with respect to the inclusions one obtains the complex $\mathcal{P}_C$.

Let F be a facette in $\mathbb{R}^n$. Denote by $G_F \subset U_{2n}(A)$ the subgroup generated by the elements $s_{ij}(a)$ where $e_i - e_j > 0$ on F, $r_{ij}(a)$ where $e_i + e_j > 0$ on F, $t_{ij}(a)$ where $e_i + e_j < 0$ on F, $p_k(b)$ where $e_k > 0$ on F, $q_k(c)$ where $e_k < 0$ on F. This is so called unipotent subgroup corresponding to the facette F.

Define the ordering on the set of pairs (α, F), where $\alpha \in U_{2n}(A)$ and F is a facet: $(\alpha', F') < (\alpha'', F'')$ iff $\alpha' G_{F'} \subset \alpha'' G_{F''}$ and $F' \subset \overline{F''}$.

Denote by $U_{2n}^{BN}(A)$ the simplicial complex with k-simplexes of form $(\alpha_0, F_0) < (\alpha_1, F_1) < \ldots < (\alpha_k, F_k)$, where $F_0, F_1, \ldots, F_k$ are facettes and $\alpha_j \in U_{2n}(A)$ for all j. The sub complex defined by the condition $\alpha_j \in EU_{2n}(A)$ is denoted by $EU_{2n}^{BN}(A)$. Denote the limit groups by $U^{BN}(A)$ и $EU^{BN}(A)$ correspondingly. One can check that

$$U^{BN}(A) = KU_1^Q(A) \times EU^{BN}(A).$$

Let us define

$$KU_n^{BN}(A) = \pi_{n-1}(U^{BN}(A)), \text{ где } n \geq 1.$$

In [1] it was shown that the functors KU_n^{BN} and KU_n^Q ($n \geq 2$) are equivalent, and moreover, there is a natural homotopy equivalence $U^{BN}(A) \cong \Omega BU(A)^+$.

2 K^D-groups and the even orthogonal group

The ideas presented in the previous section lead us to groups $K_*^D(A)$ whose construction is based on the root system D.

Consider facettes in $\mathbb{R}^n$ defined by the hyperplanes $e_i \pm e_j = 0$, $1 \leq i < j \leq n$. Denote these facettes by Φ_j to distinguish them the from the facettes defined by the root system C.

Let $\mathcal{P}_D^n$ denote the simplicial complex whose k–simplices are $(k+1)$-tuples $\Phi_0 < \Phi_1 < \ldots < \Phi_k$. For a D-facette $\Phi \subset \mathbb{R}^n$ denote by $G_\Phi \subset U_{2n}(A)$ the subgroup generates by the elements $s_{ij}(a)$ where $e_i - e_j > 0$ on Φ, $r_{ij}(a)$ where $e_i + e_j > 0$ on Φ, $t_{ij}(a)$ where $e_i + e_j < 0$ on Φ.

The set of all pairs $(\alpha; \Phi)$, where $\alpha \in U_{2n}(A)$ and Φ is a D-facet, is partially ordered by the condition that $(\alpha', \Phi') < (\alpha'', \Phi'')$ iff $\alpha' G_{\Phi'} \subseteq \alpha'' G_{\Phi''}$ and $\Phi' \subseteq \overline{\Phi''}$.

Let $U_{2n}^D(A)$ denote the simplicial complex whose k-simplices are $(\alpha_0, \Phi_0) < (\alpha_1, \Phi_1) < \ldots < (\alpha_k, \Phi_k)$ where $\Phi_0, \Phi_1, \ldots, \Phi_k$ are D-facettes and $\alpha_j \in U_{2n}(A)$. Also let $U^D(A) = \lim\limits_{\rightarrow} U_{2n}^D(A)$

Define

$$KU_n^D(A) = \pi_{n-1}(U^D(A)), \text{ где } n \geq 1.$$

Now let A be a commutative ring with 1. Let $a^* = a$, $\varepsilon = 1$, $\Lambda = \Lambda_{min} = 0$. Then the corresponding unitary group $U_{2n}(A, \varepsilon, \Lambda)$ coincides with the even orthogonal group $O_{2n}(A)$.

Theorem 1. *There exists a natural isomorphism* $K_n^D(A) = K_n^{BN}(A)$.

Remind (see. [6, 1]) that one has cartesian squares of spaces

$$\begin{array}{ccc} W_C(\alpha G_F) & \to & E(U(A)) \\ \downarrow & & \downarrow \\ W_C(A) & \to & BU(A) \end{array} \tag{1}$$

and

$$\begin{array}{ccc} W_D(\alpha G_\Phi) & \to & E(U(A)) \\ \downarrow & & \downarrow \\ W_D(A) & \to & BU(A) \end{array} \tag{2}$$

Let us describe the spaces from these diagrams. $W_C(A)$ is the realization of the simplicial space which in dimension k is the disjoint union of the spaces $(F_0 < \ldots < F_k) \times BG_{F_0}$. $W_C(\alpha G_F)$ is the realization of the simplicial space which in dimension k is the disjoint union of the spaces $((\alpha_0, F_0) < \ldots < (\alpha_k, F_k)) \times E(\alpha_0 G_{F_0})$. The definitions of $W_D(A)$ and $W_D(\alpha G_\Phi)$ are analogous. The universal covering $E(G) \to BG$ on the simplicial level is defined by the correspondence $(g_0, g_1, \ldots, g_k) \mapsto (g_0^{-1}g_1, \ldots, g_{k-1}^{-1}g_k)$. And finally $E(\alpha G_F)$ is the geometric realization of the simplicial subcomplex of $E(U(A))$ whose k-simplices are $(g_0, \ldots, g_k)$ where $g_k \in \alpha G_F$. The definition of the space $E(\alpha G_\phi)$ is analogous.

On the level of bisimplicial sets the cartesian square (1) is defined by the correspondences

$$\begin{array}{ccc} ((\alpha_0, F_0) < \ldots < (\alpha_k, F_k)); (g_0, \ldots, g_l)) & \rightarrow & (g_0, \ldots, g_l) \\ \downarrow & & \downarrow \\ (F_0 < \ldots < F_k; (g_0^{-1}g_1, \ldots, g_{l-1}^{-1}g_l)) & \rightarrow & (g_0^{-1}g_1, \ldots, g_{l-1}^{-1}g_l) \end{array}$$

and the cartesian square (2) is defined by analogous correspondences with substitution of Φ_j instead of F_j.

The spaces $E(\alpha G_F)$ and $E(\alpha G_\Phi)$ are contractible therefore one has homotopy equivalences $W_C(\alpha G_F) \simeq U^{BN}(A)$ and $W_D(\alpha G_\Phi) \simeq U^D(A)$.

Hence to compare the groups $K_*^{BN}(A)$ and $K_*^D(A)$ one could try to compare the cartesian squares (1) and (2). So to prove theorem 1 it is sufficient to prove homotopy equivalence of the lower left corners of (1) and (2), that is to establish natural homotopy equivalence of the spaces $W_C(A)$ and $W_D(A)$.

Let us remind that a sheaf X of spaces over a simplicial complex K is a collection of spaces $\{X_\sigma : \sigma \in K\}$ and maps $i_{\sigma\tau} : X_\tau \to X_\sigma$ for all $\sigma < \tau$ such that $i_{\gamma\sigma} i_{\sigma\tau} = i_{\gamma\tau}$ whenever $\gamma < \sigma < \tau$. A simplicial subdivision K' of K induces a subdivision X' of X as follows: for $\sigma' \in K'$ define $X'_{\sigma'} = X_\sigma$ where $\sigma \in K$ is the smallest simplex containing σ. If $\sigma' < \tau'$ belong to K' and σ, τ are the smallest simplices of K containing σ', τ' respectively, then $\sigma < \tau$ and we let $i_{\sigma'\tau'} = i_{\sigma\tau}$.

The realization of a sheaf X is the space $|X|$ which is obtained from the disjoint union $\coprod_{\sigma \in K} \sigma \times X_\sigma$ by identification of points (s, x) and $(s, i_{\sigma\tau}(x))$ where $s \in \sigma < \tau$ and $x \in X_\tau$. The natural map $|X'| \to |X|$ is a homeomorphism.

Obviously the spaces $W_C(A)$ and $W_D(A)$ are the realization of some sheaves over $\mathcal{P}_C$ and $\mathcal{P}_D$ respectively. Denote these sheaves by W_C and W_D respectively.

Intersections of the unit sphere with D-facettes (C-facettes) define the complex $\mathcal{Q}_D^n$ ($\mathcal{Q}_C^n$ respectively). The complexes $\mathcal{P}_D^n$ and $\mathcal{P}_C^n$ are barycentric subdivisions of $\mathcal{Q}_D^n$ and $\mathcal{Q}_C^n$ respectively. The complex $\mathcal{Q}_C^n$ is a subdivision of $\mathcal{Q}_D^n$. More precisely, a simplex of $\mathcal{Q}_D^n$ is either a simplex of $\mathcal{Q}_C^n$ or is divided into two part by one of the hyperplanes $e_k = 0$. Namely, if two hyperplanes $e_k = 0$ and $e_l = 0$ intersect the simplex of $\mathcal{Q}_D^n$ transversally then in the corresponding facette there exist points such that $e_k + e_l > 0$ and points such that $-e_k - e_l < 0$ (or points such that $e_k - e_l > 0$ and points such that $-e_k + e_l < 0$).

One can check that for a C-facette F and the smallest D-facette Φ containing F one has $G_F = G_\Phi$. Note that in general case for rings with involution this isomorphism does not hold.

Therefore there exists a common subdivision $\hat{\mathcal{P}}^n$ of complexes $\mathcal{P}_D^n$ and $\mathcal{P}_C^n$ such that, induced sheaves W'_C and W'_D over it coincide. Taking the realizations we obtain our claim.

3 Further discussion

Assume $\Lambda = \Lambda_m in \neq 0$. This case is more difficult for the following reason. For a C-facette F and the smallest D-facette Φ containing F the inclusion $G_\Phi \subset G_F$ is presumably strict for most facettes. The reason is that for $\Lambda \neq 0$ there are so called long roots unipotent $p_i(b)$ and $q_i(c)$ which are not used as generators for elementary group for the root system D. This presumably shows that one cannot expect the group generated by the short root unipotents s_{ij}, t_{ij}, r_{ij} to be perfect and coinciding with the commutant of the corresponding unitary group. There is nontrivial example even in commutative case. For a commutative ring A, trivial involution, $\varepsilon = -1$ and $\Lambda = \Lambda_{max} = A$ one obtains symplectic K-theory. This leads for the following question: what part of the symplectic K-theory can be recovered from $KU^D(A)$?

Nevertheless the following statement shows that in the case $\Lambda = \Lambda_{min}$ the difference between the root systems C and D and corresponding generators of $EU(A)$ is more subtle.

Lemma 2. *Assume $\Lambda = \Lambda_{min}$. Then the group $EU(A)$ is generated by elementary matrices s_{ij}, r_{ij}, t_{ij}.*

Proof. In $EU_{2n}(A)$ for $n \geq 2$ one has the relations

$$[s_{ij}(a), r_{ji}(1)] = p_i(a - a^*\varepsilon^*)$$
$$[s_{ji}(a), t_{ij}(\varepsilon)] = q_i(a - a^*\varepsilon)$$

which shows that even in unstable range $EU_{2n}(A)$ is generated by the short root unipotents. □

Unfortunately there are no such statement for groups like G_F. To be more precise assume that for a facette F one has $e_i - e_j > 0$ and $e_i + e_j > 0$ on F. Hence $e_i > 0$ on F. Generators s_{ij}, r_{ij}, p_i belong to G_F and one can use the relation from the proof of the previous lemma to see that p_i can be excluded from the list of generators of G_F.

On the other hand this facette F has a face F_0 defined by the equation $e_i + e_j = 0$ and one no longer can apply the relation $[s_{ij}(a), r_{ji}(1)] = p_i(a - a^*\varepsilon^*)$ because there is no generator r_{ij} in G_{F_0}.

Moreover, consider for example the facette F defined by $e_1 = e_2 = \ldots = e_n > 0$. The corresponding group (in unstable range) G_F is abelian and is generated only by the long root unipotents p_i, $i = 1, \ldots, n$, while there are no short root generators.

For a C-facette F define the group G_F^D to be the subgroup of G_F generated only by short roots which are positive on F. Clearly for $F' \subset F$ on has the inclusions

$$\begin{array}{ccc} G_F & \supset & G_{F'} \\ \cup & & \cup \\ G_F^D & \supset & G_{F'}^D \end{array} \tag{3}$$

Now consider the space $\tilde{W}_C(A)$ which is defined in the same way as $W_C(A)$ using the groups G_F^D instead of G_F. Clearly one has a map $h : \tilde{W}_C(A) \to W_C(A)$ and the map $H : \tilde{\mathcal{P}}_C \to \mathcal{P}_C$ of the corresponding simplicial sheaves.

Hence to investigate whether the map h is a homotopy equivalence it is natural to investigate the map H. For that purpose it could be useful to consider a kind of cokernel of H because the quotient G_F/G_F^D is not too big and the generator of G_F which are not in G_F^D generate an abelian subgroup in G_F.

Acknowledgements

The manuscript was finished in the framework of a project of the research group "K-Theory and Noncommutative Geometr", at VIASM. The author thanks the Institute for hospitality and support during the visit in December 2015 – January 2016.

The author thanks all the members of the research group and the participants of seminars and lectures, especially Professor Do Ngoc Diep and Professor Nguyen Le Anh for very useful discussions.

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