

Co-ordinated convexity according to a pair of quasi-arithmetic means on the rectangle from plane and inequalities

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Abstract

We consider a class of generalized convex functions, which are defined according to a pair of quasi-arithmetic means on the rectangle from the plane and called co-ordinates $(\mathcal{M}_\phi, \mathcal{M}_\psi)$ -convex functions, and establish various Fejér type inequalities for such a function class. Applications to inequalities involving the gamma function, the beta function, the fractional functions and special means are also included.

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1. Introduction

The Hermite-Hadamard inequality, name after Charles Hermite [21] and Jacques Hadamard [19] and sometimes also called Hadamard's inequality gives us a lower and an upper estimations for the integral mean value of any convex function on a closed interval, involving the midpoint and the endpoints of the domain. More precisely, if $f : [a, b] \rightarrow \mathbb{R}$ is convex, then the following chain of inequalities hold:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \quad (1.1)$$

There is an extensive literature devoted to develop applications of this inequality, as well as to discuss its extensions, by considering other measures, other kinds of convexity, or higher dimensions (see, for example, [4, 7, 9, 12, 13, 16, 17, 20, 24, 25, 29, 30, 34, 35, 45, 46, 47, 48, 49, 50, 42, 43]). Many classical results related to this inequality can be found in the

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monograph of Pečarić, Proschan and Tong [38]. Especially, in the last two decades it has received much attention. The monograph of Dragomir and Pearce [14] gives a comprehensive review of this literature.

In recent years, many mathematicians have studied the results for the inequalities for co-ordinated convex functions [14, 22, 36, 23, 42, 39]. Especially, in [10], Dragomir established the following similar inequality of Hadamard's type for co-ordinated convex mapping on a rectangle from the plane $\mathbb{R}^2$.

Theorem 1.1. *Suppose that $f : \Delta = [a, b] \times [c, d] \rightarrow \mathbb{R}$ is convex on the co-ordinates on Δ . Then one has the inequalities:*

$$\begin{aligned}
& f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \\
& \leq \frac{1}{2} \left[\frac{1}{b-a} \int_a^b f\left(x, \frac{c+d}{2}\right) dx + \frac{1}{d-c} \int_c^d f\left(\frac{a+b}{2}, y\right) dy \right] \\
& \leq \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x, y) dx dy \\
& \leq \frac{1}{4} \left[\frac{1}{b-a} \int_c^d f(x, c) dx + \frac{1}{b-a} \int_a^b f(x, d) dx \right. \\
& \quad \left. + \frac{1}{d-c} \int_c^d f(a, y) dy + \frac{1}{d-c} \int_c^d f(b, y) dy \right] \\
& \leq \frac{f(a, c) + f(a, d) + f(b, c) + f(b, d)}{4}.
\end{aligned} \tag{1.2}$$

These inequalities are sharp.

Recently, Duc, Hue, Nhan and Tuan [8] was consider a class of generalized convex functions which are defined according to a pair of quasi-arithmetic means and called $(\mathcal{M}_\phi, \mathcal{M}_\psi)$ -convex functions, and establish various Fejér type inequalities for such a function class.

In this paper, we investigate weighted quasi-arithmetic means on the rectangle from the plane.

Let $\phi : [a, b] \rightarrow \mathbb{R}$ a continuous and strictly monotonic function. The quasi-arithmetic mean of a and b with weight $\alpha \in [0, 1]$ is denoted by $\mathcal{M}_\phi(a, b; \alpha)$ and defined by [5]

$$\mathcal{M}_\phi(a, b; \alpha) = \phi^{-1}(\alpha\phi(a) + (1-\alpha)\phi(b)).$$

Here and subsequently, E and J denote open intervals in the real line $\mathbb{R}$, $\phi : E \rightarrow \mathbb{R}$ and $\psi : J \rightarrow \mathbb{R}$ are continuous and strictly monotonic functions. With the quasi-arithmetic means $\mathcal{M}_\phi$ and $\mathcal{M}_\psi$ in hand, we are now in a position to generalize the notion of convexity. According to Aumann [2] (see also [33]), a function $f : E \rightarrow J$ is said to be $(\mathcal{M}_\phi, \mathcal{M}_\psi)$ -convex if it verifies the following analogue of Jensen's inequality:

$$f(\mathcal{M}_\phi(x, y; \alpha)) \leq \mathcal{M}_\psi(f(x), f(y); \alpha) \tag{1.3}$$

for all $x, y \in E$ and $\alpha \in [0, 1]$.

Let us consider the bidimensional interval $I := [a_1, b_1] \times [a_2, b_2]$ in $\mathbb{R}^2$ with $a_1 < b_1, a_2 < b_2$ and $[a_1, b_1] \subset E, [a_2, b_2] \subset E$, E and J are open intervals in the real line $\mathbb{R}$, $\phi : E \rightarrow \mathbb{R}$ and $\psi : J \rightarrow \mathbb{R}$ are continuous and strictly monotonic functions.

A function $f : I \rightarrow J$ is co-ordinated $(\mathcal{M}_\phi, \mathcal{M}_\psi)$ -convex on I if the partial mappings

$$f_y : [a_1, b_1] \rightarrow J, \quad f_y(u) := f(u, y)$$

and

$$f_x : [a_2, b_2] \rightarrow J, \quad f_x(v) := f(x, v)$$

are $(\mathcal{M}_\phi, \mathcal{M}_\psi)$ -convex for all $y \in [a_1, b_1]$ and $x \in [a_2, b_2]$.

Note that every $(\mathcal{M}_\phi, \mathcal{M}_\psi)$ -convex mapping $f : I \rightarrow J$ is co-ordinated $(\mathcal{M}_\phi, \mathcal{M}_\psi)$ -convex but the converses not generally true.

Accordingly, the aim of the present paper is to deal with interpolating inequalities of Fejér, which not only provide a natural and intrinsic characterization of the co-ordinated $(\mathcal{M}_\phi, \mathcal{M}_\psi)$ -convex functions, but also agree with a wide class of known inequalities of Hermite-Hadamard and Fejér type for different kinds of convexity. At the same time, we establish some inequalities involving the gamma function, the beta function, the continuous probability density functions.

2. Fejér type inequalities for co-ordinates $(\mathcal{M}_\phi, \mathcal{M}_\psi)$ -convex functions

In what follows, let $f : I \rightarrow J$ be co-ordinated $(\mathcal{M}_\phi, \mathcal{M}_\psi)$ -convex on I , and let $\omega_i, \theta_i : [0, 1] \rightarrow [0, \infty)$ be integrable, with $\int_0^s \omega_i(t)dt > 0; \int_s^1 \theta_i(t)dt > 0, i = 1, 2$ for all $s \in (0, 1)$.

For simplicity of notation, we will write

$$\begin{aligned} a &= (a_1, a_2), b = (b_1, b_2), t = (t_1, t_2) \in [0, 1]^2, \\ \mathcal{M}_\phi(a, b; \alpha) &= (\mathcal{M}_\phi(a_1, b_1; \alpha), \mathcal{M}_\phi(a_2, b_2; \alpha)), \\ \mathcal{L}_1(t_1) &= \mathcal{M}_\phi(a_1, \mathcal{M}_\phi(a_1, b_1; \alpha); t_1), \\ \mathcal{L}_2(t_2) &= \mathcal{M}_\phi(a_2, \mathcal{M}_\phi(a_2, b_2; \alpha); t_2), \\ \mathcal{L}(t) &= \mathcal{L}(t_1, t_2) = \left(\mathcal{L}_1(t_1); \mathcal{L}_2(t_2) \right), \\ \mathcal{R}_1(t_1) &= \mathcal{M}_\phi(b_1, \mathcal{M}_\phi(a_1, b_1; \alpha); t_1), \\ \mathcal{R}_2(t_2) &= \mathcal{M}_\phi(b_2, \mathcal{M}_\phi(a_2, b_2; \alpha); t_2), \\ \mathcal{R}(t) &= \mathcal{R}(t_1, t_2) = \left(\mathcal{R}_1(t_1); \mathcal{R}_2(t_2) \right). \end{aligned}$$

We see that

$$\begin{aligned} \mathcal{L}(1, 1) &= (a_1, a_2), & \mathcal{R}(1, 1) &= (b_1, b_2), \\ \mathcal{L}(1, 0) &= (a_1, \mathcal{M}_\phi(a_2, b_2; \alpha)), & \mathcal{R}(1, 0) &= (b_1, \mathcal{M}_\phi(a_2, b_2; \alpha)), \\ \mathcal{L}(0, 1) &= (\mathcal{M}_\phi(a_1, b_1; \alpha), a_2), & \mathcal{R}(0, 1) &= (\mathcal{M}_\phi(a_1, b_1; \alpha), b_2), \\ \mathcal{L}(0, 0) &= (\mathcal{M}_\phi(a_1, b_1; \alpha), \mathcal{M}_\phi(a_2, b_2; \alpha)), & \mathcal{R}(0, 0) &= (\mathcal{M}_\phi(a_1, b_1; \alpha), \mathcal{M}_\phi(a_2, b_2; \alpha)). \end{aligned}$$

Theorem 2.1. Let $\mathcal{F}, \mathcal{G} : [0, 1]^2 \rightarrow \mathbb{R}$ be defined by

$$\begin{aligned} \mathcal{F}(t) = \psi^{-1} & \left[\alpha^2 \psi \circ f(\mathcal{L}_1(t_1), \mathcal{L}_2(t_2)) + \alpha(1 - \alpha) \psi \circ f(\mathcal{L}_1(t_1), \mathcal{R}_2(t_2)) \right. \\ & \left. + (1 - \alpha) \alpha \psi \circ f(\mathcal{R}_1(t_1), \mathcal{L}_2(t_2)) + (1 - \alpha)^2 \psi \circ f(\mathcal{R}_1(t_1), \mathcal{R}_2(t_2)) \right] \end{aligned}$$

and

$$\begin{aligned} \mathcal{G}(t) = \psi^{-1} & \left[t_1 t_2 \psi \circ \mathcal{F}(1, 1) + t_1(1 - t_2) \psi \circ \mathcal{F}(1, 0) \right. \\ & \left. + (1 - t_1) t_2 \psi \circ \mathcal{F}(0, 1) + (1 - t_1)(1 - t_2) \psi \circ \mathcal{F}(0, 0) \right] \end{aligned}$$

respectively.

1. The functions $\mathcal{F}$ and $\mathcal{G}$ are $\mathcal{M}_\psi$ -convex, increasing on co-ordinates on $[0, 1]^2$, and

$$\begin{aligned} \mathcal{F}(0, 0) &= \mathcal{G}(0, 0) = f(\mathcal{M}_\phi(a, b; \alpha)), \\ \mathcal{F}(1, 0) &= \mathcal{G}(1, 0) = \mathcal{M}_\psi[f(a_1, \mathcal{M}_\phi(a_2, b_2; \alpha)), f(b_1, \mathcal{M}_\phi(a_2, b_2; \alpha)); \alpha], \\ \mathcal{F}(0, 1) &= \mathcal{G}(0, 1) = \mathcal{M}_\psi[f(\mathcal{M}_\phi(a_1, b_1; \alpha), a_2), f(\mathcal{M}_\phi(a_1, b_1; \alpha), b_2); \alpha], \\ \mathcal{F}(t) &\leq \mathcal{G}(t), \quad t = (t_1, t_2) \in (0, 1)^2, \\ \mathcal{F}(1, 1) &= \mathcal{G}(1, 1) = \psi^{-1} \left[\alpha^2 \psi \circ f(a_1, a_2) + \alpha(1 - \alpha) \psi \circ f(a_1, b_2) \right. \\ &\quad \left. + (1 - \alpha) \alpha \psi \circ f(b_1, a_2) + (1 - \alpha)^2 \psi \circ f(b_1, b_2) \right]. \end{aligned} \tag{2.1}$$

2. For $s = (s_1, s_2) \in (0, 1]^2$, define

$$\begin{aligned} \mathcal{I}(s) &= \psi^{-1} \left(\frac{\int_0^{s_1} \int_0^{s_2} \psi \circ \mathcal{F}(t_1, t_2) w_1(t_1) w_2(t_2) dt_1 dt_2}{\int_0^{s_1} w_1(t_1) dt_1 \int_0^{s_2} w_2(t_2) dt_2} \right), \\ \beta(s) &= (\beta_1(s_1), \beta_2(s_2)) = \left(\frac{\int_0^{s_1} t_1 w_1(t_1) dt_1}{\int_0^{s_1} w_1(t_1) dt_1}, \frac{\int_0^{s_2} t_2 w_2(t_2) dt_2}{\int_0^{s_2} w_2(t_2) dt_2} \right), \end{aligned}$$

and

$$\mathcal{H}(s) = \psi^{-1} \left(\frac{\int_0^{s_2} \psi \circ \mathcal{G}(\beta_1(s_1), t_2) w_2(t_2) dt_2}{2 \int_0^{s_2} w_2(t_2) dt_2} + \frac{\int_0^{s_1} \psi \circ \mathcal{G}(t_1, \beta_2(s_2)) w_1(t_1) dt_1}{2 \int_0^{s_1} w_1(t_1) dt_1} \right).$$

Then $\mathcal{F} \circ \beta$, $\mathcal{I}$ and $\mathcal{G} \circ \beta$ increase on co-ordinates on $(0, 1]^2$ and satisfy

$$\begin{aligned} \lim_{s_1 \rightarrow 0^+} \lim_{s_2 \rightarrow 0^+} \mathcal{F} \circ \beta(s) &= \lim_{s_1 \rightarrow 0^+} \lim_{s_2 \rightarrow 0^+} \mathcal{I}(s) = \lim_{s_1 \rightarrow 0^+} \lim_{s_2 \rightarrow 0^+} \mathcal{G} \circ \beta(s) = f(\mathcal{M}_\phi(a, b; \alpha)), \\ \mathcal{F} \circ \beta(s) &\leq \mathcal{I}(s) \leq \mathcal{H}(s) \leq \mathcal{G} \circ \beta(s) \leq \mathcal{F}(s), \quad s = (s_1, s_2) \in (0, 1]^2. \end{aligned} \tag{2.2}$$

3. For $s = (s_1, s_2) \in (0, 1]^2$, define

$$\mathcal{J}(s) = \psi^{-1} \left(\frac{\int_{s_1}^1 \int_{s_2}^1 \psi \circ \mathcal{F}(t_1, t_2) \theta_1(t_1) \theta_2(t_2) dt_1 dt_2}{\int_{s_1}^1 \theta_1(t_1) dt_1 \int_{s_2}^1 \theta_2(t_2) dt_2} \right),$$

$$\gamma(s) = (\gamma_1(s_1), \gamma_2(s_2)) = \left(\frac{\int_{s_1}^1 t_1 \theta_1(t_1) dt_1}{\int_{s_1}^1 \theta_1(t_1) dt_1}, \frac{\int_{s_2}^1 t_2 \theta_2(t_2) dt_2}{\int_{s_2}^1 \theta_2(t_2) dt_2} \right),$$

and

$$\mathcal{K}(s) = \psi^{-1} \left(\frac{\int_{s_2}^1 \psi \circ \mathcal{G}(\gamma_1(s_1), t_2) w_2(t_2) dt_2}{2 \int_{s_2}^1 w_2(t_2) dt_2} + \frac{\int_{s_1}^1 \psi \circ \mathcal{G}(t_1, \gamma_2(s_2)) w_1(t_1) dt_1}{2 \int_{s_1}^1 w_1(t_1) dt_1} \right).$$

Then $\mathcal{F} \circ \gamma$, $\mathcal{J}$ and $\mathcal{G} \circ \gamma$ increase on co-ordinates on $[0, 1]^2$ and satisfy

$$\mathcal{G}(s) \leq \mathcal{F} \circ \gamma(s) \leq \mathcal{J}(s) \leq \mathcal{K}(s) \leq \mathcal{G} \circ \gamma(s), \quad s = (s_1, s_2) \in (0, 1]^2,$$

$$\lim_{s_1 \rightarrow 1^-} \lim_{s_2 \rightarrow 1^-} \mathcal{F} \circ \gamma(s) = \lim_{s_1 \rightarrow 1^-} \lim_{s_2 \rightarrow 1^-} \mathcal{J}(s) = \lim_{s_1 \rightarrow 1^-} \lim_{s_2 \rightarrow 1^-} \mathcal{G} \circ \gamma(s) = \mathcal{G}(1, 1). \quad (2.3)$$

If, in addition, $w_i = \theta_i, i = 1, 2$, then $\mathcal{I}(1, 1) = \mathcal{J}(0, 0)$.

The following three lemmas that will be imperative to the proof of the main result of the above theory.

The first lemma, called Aczél correspondence principle [1] (see also [33, Lemma A.2.2]), reduces the co-ordinates $(\mathcal{M}_\phi, \mathcal{M}_\psi)$ -convexity to the usual convexity of a function derived via a change of variable and a change of function.

Lemma 2.2 (Aczél correspondence principle). *If ψ is increasing on J , then f is co-ordinates $(\mathcal{M}_\phi, \mathcal{M}_\psi)$ -convex on I if and only if $\psi \circ f(\cdot, \phi^{-1}(y))$ is convex on $\phi([a_1, b_1])$ and $\psi \circ f(\phi^{-1}(x), \cdot)$ is convex on $\phi([a_2, b_2])$. Conversely, if ψ is decreasing on J , then f is co-ordinates $(\mathcal{M}_\phi, \mathcal{M}_\psi)$ -convex on I if and only if $\psi \circ f(\cdot, \phi^{-1}(y))$ is concave on $\phi([a_1, b_1])$ and $\psi \circ f(\phi^{-1}(x), \cdot)$ is concave on $\phi([a_2, b_2])$.*

The following lemmas [8] provides a useful inequality related to convex functions, which generalizes the result of Hwang, Tseng and Yang given in [22, Lemma].

Lemma 2.3 ([8]). *Let $H : [A, B] \subset \mathbb{R} \rightarrow \mathbb{R}$ be a convex function and let $\beta \in [0, 1]$. Then for any $C, D \in [A, B]$, with $\beta A + (1 - \beta)B = \beta C + (1 - \beta)D$, one has*

$$\beta H(C) + (1 - \beta)H(D) \leq \beta H(A) + (1 - \beta)H(B). \quad (2.4)$$

Lemma 2.4 ([8]). *Let $P : [0, 1] \rightarrow \mathbb{R}$ be continuous and increasing.*

1. For $s \in (0, 1]$, define

$$P_1(s) = \frac{\int_0^s P(t) w_1(t) dt}{\int_0^s w_1(t) dt}.$$

Then P_1 is increasing on $(0, 1]$, with

$$\lim_{s \rightarrow 0^+} P_1(s) = P(0) \leq P_1(s) \leq P(s), \quad s \in (0, 1]. \quad (2.5)$$

2. Similarly, for $s \in [0, 1)$, define

$$P_2(s) = \frac{\int_s^1 P(t)w_2(t)dt}{\int_s^1 w_2(t)dt}.$$

Then P_2 is increasing on $[0, 1)$, with

$$P(s) \leq P_2(s) \leq P(1) = \lim_{s \rightarrow 1^-} P_2(s), \quad s \in [0, 1).$$

From the above lemma we can construct the following lemma that it allows one to establish various weighted interpolating inequalities for a continuous and monotonic function on co-ordinates.

Lemma 2.5. *Let $P : [0, 1]^2 \rightarrow \mathbb{R}$ be continuous and increasing on co-ordinates.*

1. For $s = (s_1, s_2) \in (0, 1]^2$, define

$$P_1(s) = \frac{\int_0^{s_1} \int_0^{s_2} P(t_1, t_2)w_1(t_1)w_2(t_2)dt_1dt_2}{\int_0^{s_1} w_1(t_1)dt_1 \int_0^{s_2} w_2(t_2)dt_2}.$$

Then P_1 is increasing on co-ordinates on $(0, 1]^2$, with

$$\lim_{s_1 \rightarrow 0^+} \lim_{s_2 \rightarrow 0^+} P_1(s) = P(0, 0) \leq P_1(s) \leq P(s), \quad s = (s_1, s_2) \in (0, 1]^2. \quad (2.6)$$

2. Similarly, for $s \in [0, 1)^2$, define

$$P_2(s) = \frac{\int_{s_1}^1 \int_{s_2}^1 P(t_1, t_2)\theta_1(t_1)\theta_2(t_2)dt_1dt_2}{\int_{s_1}^1 \theta_1(t_1)dt_1 \int_{s_2}^1 \theta_2(t_2)dt_2}.$$

Then P_2 is increasing on co-ordinates on $[0, 1)^2$, with

$$P(s) \leq P_2(s) \leq P(1, 1) = \lim_{s_1 \rightarrow 1^-} \lim_{s_2 \rightarrow 1^-} P_2(s), \quad s = (s_1, s_2) \in [0, 1)^2.$$

We are now in a position to prove the theorem.

Proof of Theorem 2.1. Since ψ is strictly monotonic, we need to examine two possibilities of ψ . Assume first that ψ is strictly increasing on J . But then, because ψ is also continuous on J , ψ^{-1} is continuous and strictly increasing on $\psi(J)$. Furthermore, by Aczél correspondence principle, $\psi \circ f(\cdot, \phi^{-1}(y))$ is convex on $\phi([a_1, b_1])$ and $\psi \circ f(\phi^{-1}(x), \cdot)$ is convex on $\phi([a_2, b_2])$.

1. To show $\mathcal{F}$ is co-ordinates $\mathcal{M}_\psi$ -convex on $[0, 1]^2$, it suffices to show that $\psi \circ \mathcal{F}$ is co-ordinates convex on $[0, 1]^2$. We have

$$\begin{aligned} \psi \circ \mathcal{F}(t) &= \alpha^2 \psi \circ f\left(\phi^{-1}(A(t_1)), \phi^{-1}(A(t_2))\right) \\ &\quad + \alpha(1 - \alpha) \psi \circ f\left(\phi^{-1}(A(t_1)), \phi^{-1}(B(t_2))\right) \\ &\quad + (1 - \alpha)\alpha \psi \circ f\left(\phi^{-1}(B(t_1)), \phi^{-1}(A(t_2))\right) \\ &\quad + (1 - \alpha)^2 \psi \circ f\left(\phi^{-1}(B(t_1)), \phi^{-1}(B(t_2))\right), \end{aligned}$$

where

$$A(t_i) = t_i\phi(a_i) + (1 - t_i)(\alpha\phi(a_i) + (1 - \alpha)\phi(b_i)), \quad i = 1, 2, \quad (2.7)$$

and

$$B(t_i) = t_i\phi(b_i) + (1 - t_i)(\alpha\phi(a_i) + (1 - \alpha)\phi(b_i)), \quad i = 1, 2. \quad (2.8)$$

Since $\psi \circ f \circ \phi^{-1}(t_1, \cdot)$ is convex on $\phi([a_1, b_1])$, $A(t_1)$ and $B(t_1)$ are linear on $[0, 1]$, it follows that $\psi \circ \mathcal{F}(t_1, \cdot)$ is convex on $[0, 1]$ as claimed. Similarly, $\psi \circ \mathcal{F}(\cdot, t_2)$ is convex on $[0, 1]$. The co-ordinates $\mathcal{M}_\psi$ -convexity of $\mathcal{G}$ on $[0, 1]^2$ immediately follows from the definition of $\mathcal{G}$.

Next, it is easily seen that

$$\begin{aligned} \mathcal{F}(1, 1) &= \mathcal{G}(1, 1), \\ \mathcal{F}(0, 0) &= \mathcal{G}(0, 0) = f(\mathcal{M}_\phi(a, b; \alpha)), \\ \mathcal{F}(1, 0) &= \mathcal{G}(1, 0) = \mathcal{M}_\psi[f(a_1, \mathcal{M}_\phi(a_2, b_2; \alpha)), f(b_1, \mathcal{M}_\phi(a_2, b_2; \alpha))], \\ \mathcal{F}(0, 1) &= \mathcal{F}(0, 1) = \mathcal{M}_\psi[f(\mathcal{M}_\phi(a_1, b_1; \alpha), a_2), f(\mathcal{M}_\phi(a_1, b_1; \alpha), b_2)]. \end{aligned}$$

Now, by the convexity of $\psi \circ f(\phi^{-1}(x), \cdot)$ and $\psi \circ f(\cdot, \phi^{-1}(y))$,

$$\begin{aligned} &\psi \circ f(\phi^{-1}(A(t_1)), \phi^{-1}(A(t_2))) \\ &\leq t_1\psi \circ f(a_1, \phi^{-1}(A(t_2))) + (1 - t_1)\psi \circ f(\mathcal{M}_\phi(a_1, b_1; \alpha), \phi^{-1}(A(t_2))) \\ &\leq t_1 \left(t_2\psi \circ f(a_1, a_2) + (1 - t_2)\psi \circ f(a_1, \mathcal{M}_\phi(a_2, b_2; \alpha)) \right) \\ &\quad + (1 - t_1) \left(t_2\psi \circ f(\mathcal{M}_\phi(a_1, b_1; \alpha), a_2) + (1 - t_2)\psi \circ f(\mathcal{M}_\phi(a_1, b_1; \alpha), \mathcal{M}_\phi(a_2, b_2; \alpha)) \right), \end{aligned}$$

$$\begin{aligned} &\psi \circ f(\phi^{-1}(A(t_1)), \phi^{-1}(B(t_2))) \\ &\leq t_1\psi \circ f(a_1, \phi^{-1}(B(t_2))) + (1 - t_1)\psi \circ f(\mathcal{M}_\phi(a_1, b_1; \alpha), \phi^{-1}(B(t_2))) \\ &\leq t_1 \left(t_2\psi \circ f(a_1, b_2) + (1 - t_2)\psi \circ f(a_1, \mathcal{M}_\phi(a_2, b_2; \alpha)) \right) \\ &\quad + (1 - t_1) \left(t_2\psi \circ f(\mathcal{M}_\phi(a_1, b_1; \alpha), b_2) + (1 - t_2)\psi \circ f(\mathcal{M}_\phi(a_1, b_1; \alpha), \mathcal{M}_\phi(a_2, b_2; \alpha)) \right), \end{aligned}$$

and

$$\begin{aligned} &\psi \circ f(\phi^{-1}(B(t_1)), \phi^{-1}(A(t_2))) \\ &\leq t_1\psi \circ f(b_1, A(t_2)) + (1 - t_1)\psi \circ f(\mathcal{M}_\phi(a_1, b_1; \alpha), A(t_2)) \\ &\leq t_1 \left(t_2\psi \circ f(b_1, a_2) + (1 - t_2)\psi \circ f(b_1, \mathcal{M}_\phi(a_2, b_2; \alpha)) \right) \\ &\quad + (1 - t_1) \left(t_2\psi \circ f(\mathcal{M}_\phi(a_1, b_1; \alpha), a_2) + (1 - t_2)\psi \circ f(\mathcal{M}_\phi(a_1, b_1; \alpha), \mathcal{M}_\phi(a_2, b_2; \alpha)) \right), \end{aligned}$$

$$\begin{aligned}
& \psi \circ f(\phi^{-1}(B(t_1)), \phi^{-1}(B(t_2))) \\
& \leq t_1 \psi \circ f(b_1, B(t_2)) + (1 - t_1) \psi \circ f(\mathcal{M}_\phi(a_1, b_1; \alpha), B(t_2)) \\
& \leq t_1 \left(t_2 \psi \circ f(b_1, b_2) + (1 - t_2) \psi \circ f(b_1, \mathcal{M}_\phi(a_2, b_2; \alpha)) \right) \\
& \quad + (1 - t_1) \left(t_2 \psi \circ f(\mathcal{M}_\phi(a_1, b_1; \alpha), b_2) + (1 - t_2) \psi \circ f(\mathcal{M}_\phi(a_1, b_1; \alpha), \mathcal{M}_\phi(a_2, b_2; \alpha)) \right).
\end{aligned}$$

We thus get

$$\begin{aligned}
\psi \circ \mathcal{F}(t) & \leq t_1 t_2 \left(\alpha^2 \psi \circ f(a_1, a_2) + \alpha(1 - \alpha) \psi \circ f(a_1, b_2) \right. \\
& \quad \left. + (1 - \alpha) \alpha \psi \circ f(b_1, a_2) + (1 - \alpha)^2 \psi \circ f(b_1, b_2) \right) \\
& \quad + t_1(1 - t_2) \left(\alpha \psi \circ f(a_1, \mathcal{M}_\phi(a_2, b_2; \alpha)) + (1 - \alpha) \psi \circ f(b_1, \mathcal{M}_\phi(a_2, b_2; \alpha)) \right) \\
& \quad + t_2(1 - t_1) \left(\alpha \psi \circ f(\mathcal{M}_\phi(a_1, b_1; \alpha), a_2) + (1 - \alpha) \psi \circ f(\mathcal{M}_\phi(a_1, b_1; \alpha), b_2) \right) \\
& \quad + (1 - t_1)(1 - t_2) \psi \circ f(\mathcal{M}_\phi(a_1, b_1; \alpha), \mathcal{M}_\phi(a_2, b_2; \alpha)) \\
& = \psi \circ \mathcal{G}(t)
\end{aligned}$$

and, because ψ^{-1} is increasing on $\psi(J)$,

$$\mathcal{F}(t) \leq \mathcal{G}(t), \quad t = (t_1, t_2) \in (0, 1]^2,$$

whence (2.1) is verified.

We proceed to show that $\mathcal{F}$ is increasing co-ordinates on $[0, 1]^2$. To this end, suppose that $0 < t_i < r_i \leq 1, i = 1, 2$. By the co-ordinates $(\mathcal{M}_\phi, \mathcal{M}_\psi)$ -convexity of f ,

$$\begin{aligned}
\mathcal{F}(0, t_2) & = \psi^{-1} \left[\alpha^2 \psi \circ f(\mathcal{M}_\phi(a_1, b_1; \alpha), \mathcal{L}_2(t_2)) + \alpha(1 - \alpha) \psi \circ f(\mathcal{M}_\phi(a_1, b_1; \alpha), \mathcal{R}_2(t_2)) \right. \\
& \quad \left. + (1 - \alpha) \alpha \psi \circ f(\mathcal{M}_\phi(a_1, b_1; \alpha), \mathcal{L}_2(t_2)) + (1 - \alpha)^2 \psi \circ f(\mathcal{M}_\phi(a_1, b_1; \alpha), \mathcal{R}_2(t_2)) \right] \\
& = \psi^{-1} \left[\alpha \psi \circ f(\mathcal{M}_\phi(a_1, b_1; \alpha), \mathcal{L}_2(t_2)) + (1 - \alpha) \psi \circ f(\mathcal{M}_\phi(a_1, b_1; \alpha), \mathcal{R}_2(t_2)) \right] \\
& \leq \psi^{-1} \left[\alpha \psi \circ \mathcal{M}_\psi[f(\mathcal{L}_1(t_1)), \mathcal{L}_2(t_2), f(\mathcal{R}_1(t_1)), \mathcal{L}_2(t_2)]; \alpha] \right. \\
& \quad \left. + (1 - \alpha) \psi \circ \mathcal{M}_\psi[f(\mathcal{L}_1(t_1)), \mathcal{L}_2(t_2), f(\mathcal{R}_1(t_1)), \mathcal{R}_2(t_2)]; \alpha] \right] \\
& = \mathcal{F}(t_1, t_2)
\end{aligned}$$

which yields

$$\psi \circ \mathcal{F}(t_1, t_2) \geq \psi \circ \mathcal{F}(0, t_2).$$

Similarly,

$$\psi \circ \mathcal{F}(t_1, t_2) \geq \psi \circ \mathcal{F}(t_1, 0).$$

Together with the co-ordinates convexity of $\psi \circ \mathcal{F}$, this gives

$$\frac{\psi \circ \mathcal{F}(r_1, t_2) - \psi \circ \mathcal{F}(t_1, t_2)}{r_1 - t_1} \geq \frac{\psi \circ \mathcal{F}(t_1, t_2) - \psi \circ \mathcal{F}(0, t_2)}{t_1 - 0} \geq 0$$

and

$$\frac{\psi \circ \mathcal{F}(t_1, r_2) - \psi \circ \mathcal{F}(t_1, t_2)}{r_2 - t_2} \geq \frac{\psi \circ \mathcal{F}(t_1, t_2) - \psi \circ \mathcal{F}(t_1, 0)}{t_2 - 0} \geq 0,$$

which implies that $\psi \circ \mathcal{F}$ is increasing co-ordinates on $[0, 1]^2$. Since ψ^{-1} is increasing on $\psi(J)$, we conclude that $\mathcal{F}$ is increasing co-ordinates on $[0, 1]^2$ as desired. Since

$$\begin{aligned} \psi \circ \mathcal{G}(t) &= t_1 \left(t_2 \psi \circ \mathcal{F}(1, 1) + (1 - t_2) \psi \circ \mathcal{F}(1, 0) \right) \\ &\quad + (1 - t_1) \left(t_2 \psi \circ \mathcal{F}(0, 1) + (1 - t_2) \psi \circ \mathcal{F}(0, 0) \right), \\ \psi \circ \mathcal{G}(t) &= t_2 \left(t_1 \psi \circ \mathcal{F}(1, 1) + (1 - t_1) \psi \circ \mathcal{F}(0, 1) \right) \\ &\quad + (1 - t_2) \left(t_1 \psi \circ \mathcal{F}(1, 0) + (1 - t_1) \psi \circ \mathcal{F}(0, 0) \right) \end{aligned}$$

and

$$\begin{aligned} \mathcal{F}(0, 1) &\leq \mathcal{F}(1, 1), \quad \mathcal{F}(1, 0) \leq \mathcal{F}(1, 1), \\ \mathcal{F}(0, 0) &\leq \mathcal{F}(0, 1), \quad \mathcal{F}(0, 0) \leq \mathcal{F}(1, 0) \end{aligned}$$

it follows that $\psi \circ \mathcal{G}$, and so does $\mathcal{G}$, increases co-ordinates on $[0, 1]^2$.

2. Applying Lemma 2.4 for $P = \psi \circ \mathcal{F}$, we conclude that $\psi \circ \mathcal{I}$ is increasing co-ordinates on $(0, 1]^2$, with

$$\lim_{s_1 \rightarrow 0^+} \lim_{s_2 \rightarrow 0^+} \psi \circ \mathcal{I}(s) = \psi \circ \mathcal{F}(0, 0) = \psi \circ f(\mathcal{M}_\phi(a, b; \alpha)).$$

Since ψ^{-1} is continuous and strictly increasing on $\psi(J)$, it follows that $\mathcal{I}$ is increasing co-ordinates on $(0, 1]^2$ and

$$\lim_{s_1 \rightarrow 0^+} \lim_{s_2 \rightarrow 0^+} \mathcal{I}(s) = f(\mathcal{M}_\phi(a, b; \alpha)).$$

Again, by Lemma 2.4, β is increasing co-ordinates on $(0, 1]^2$, with

$$\lim_{s_1 \rightarrow 0^+} \lim_{s_2 \rightarrow 0^+} \beta(s) = (0, 0) \leq \beta(s) \leq s, \quad s \in (0, 1]^2.$$

Thus, the first part of the theorem asserts that $\mathcal{F} \circ \beta$ and $\mathcal{G} \circ \beta$ are well-defined, increasing co-ordinates on $(0, 1]^2$ and

$$\lim_{s_1 \rightarrow 0^+} \lim_{s_2 \rightarrow 0^+} \mathcal{F} \circ \beta(s) = \lim_{s_1 \rightarrow 0^+} \lim_{s_2 \rightarrow 0^+} \mathcal{G} \circ \beta(s) = f(\mathcal{M}_\phi(a, b; \alpha)).$$

Our next goal is to show the inequalities in (2.2). Fix $s = (s_1, s_2) \in (0, 1]^2$. Applying Jensen's inequality (see, for example, [38, Chapter 2]) to the convex function $\psi \circ \mathcal{F}_{t_i}(t_j)$ on the interval $[0, s_j]$ with respect to the measure $w_j(t_j)dt_j$, $i, j = 1, 2$ we obtain

$$\psi \circ \mathcal{F} \left(\frac{\int_0^{s_1} t_1 w_1(t_1) dt_1}{\int_0^{s_1} w_1(t_1) dt_1}, \frac{\int_0^{s_2} t_2 w_2(t_2) dt_2}{\int_0^{s_2} w_2(t_2) dt_2} \right) \leq \frac{\int_0^{s_1} \int_0^{s_2} \psi \circ \mathcal{F}(t_1, t_2) w_1(t_1) w_2(t_2) dt_1 dt_2}{\int_0^{s_1} w_1(t_1) dt_1 \int_0^{s_2} w_2(t_2) dt_2},$$

which yields

$$\mathcal{F} \circ \beta(s) \leq \mathcal{I}(s).$$

Sine $\mathcal{F}(t_1, t_2) \leq \mathcal{G}(t_1, t_2)$ for all $t_2 \in [0, s_2]$ we get that

$$\frac{\int_0^{s_1} \psi \circ \mathcal{F}(t_1, t_2) w_1(t_1) dt_1}{\int_0^{s_1} w_1(t_1) dt_1} \leq \frac{\int_0^{s_1} \psi \circ \mathcal{G}(t_1, t_2) w_1(t_1) dt_1}{\int_0^{s_1} w_1(t_1) dt_1} \leq \psi \circ \mathcal{G}(\beta_1(s_1), t_2).$$

Hence

$$\frac{\int_0^{s_1} \int_0^{s_2} \psi \circ \mathcal{F}(t_1, t_2) w_1(t_1) w_2(t_2) dt_1 dt_2}{\int_0^{s_1} w_1(t_1) dt_1 \int_0^{s_2} w_2(t_2) dt_2} \leq \frac{\int_0^{s_2} \psi \circ \mathcal{G}(\beta_1(s_1), t_2) w_2(t_2) dt_2}{\int_0^{s_2} w_2(t_2) dt_2}. \quad (2.9)$$

In a similar way we get

$$\frac{\int_0^{s_1} \int_0^{s_2} \psi \circ \mathcal{F}(t_1, t_2) w_1(t_1) w_2(t_2) dt_1 dt_2}{\int_0^{s_1} w_1(t_1) dt_1 \int_0^{s_2} w_2(t_2) dt_2} \leq \frac{\int_0^{s_1} \psi \circ \mathcal{G}(t_1, \beta_2(s_2)) w_1(t_1) dt_1}{\int_0^{s_1} w_1(t_1) dt_1}. \quad (2.10)$$

Summing (2.9) and (2.10) we obtain

$$\begin{aligned} & \frac{\int_0^{s_1} \int_0^{s_2} \psi \circ \mathcal{F}(t_1, t_2) w_1(t_1) w_2(t_2) dt_1 dt_2}{\int_0^{s_1} w_1(t_1) dt_1 \int_0^{s_2} w_2(t_2) dt_2} \\ & \leq \frac{1}{2} \left(\frac{\int_0^{s_2} \psi \circ \mathcal{G}(\beta_1(s_1), t_2) w_2(t_2) dt_2}{\int_0^{s_2} w_2(t_2) dt_2} + \frac{\int_0^{s_1} \psi \circ \mathcal{G}(t_1, \beta_2(s_2)) w_1(t_1) dt_1}{\int_0^{s_1} w_1(t_1) dt_1} \right), \end{aligned}$$

which, as the function ψ^{-1} is increasing, implies

$$\mathcal{I}(s) \leq \psi^{-1} \left(\frac{\int_0^{s_2} \psi \circ \mathcal{G}(\beta_1(s_1), t_2) w_2(t_2) dt_2}{2 \int_0^{s_2} w_2(t_2) dt_2} + \frac{\int_0^{s_1} \psi \circ \mathcal{G}(t_1, \beta_2(s_2)) w_1(t_1) dt_1}{2 \int_0^{s_1} w_1(t_1) dt_1} \right) = \mathcal{H}(s).$$

Next, it is easily seen that

$$\mathcal{H}(s) \leq \mathcal{G} \circ \beta(s).$$

It remains to show

$$\mathcal{G} \circ \beta(s) \leq \mathcal{F}(s).$$

We utilize Lemma 2.3, with $H = H(x, y) = \psi \circ f(\phi^{-1}(x), \phi^{-1}(y))$, $A = \min\{A(s_1), B(s_1)\}$, $B = \max\{A(s_1), B(s_1)\}$, $C = \min\{A(\beta_1(s_1)), B(\beta_1(s_1))\}$, $D = \max\{A(\beta_1(s_1)), B(\beta_1(s_1))\}$, and

$$\beta = \begin{cases} \alpha & \text{if } A(s_1) \leq B(s_1), \\ 1 - \alpha & \text{if } A(s_1) > B(s_1), \end{cases}$$

where $A(\cdot)$ and $B(\cdot)$ are as in (2.7) and (2.8), respectively. To do this, we need to ensure that $C, D \in [A, B]$, with $\beta A + (1 - \beta)B = \beta C + (1 - \beta)D$. But this immediately follows from the fact that

$$\beta A + (1 - \beta)B = \beta C + (1 - \beta)D = \alpha \phi(a) + (1 - \alpha)\phi(b)$$

and

$$B - A = s_1|\phi(b) - \phi(a)| \geq \beta_1(s_1)|\phi(b) - \phi(a)| = D - C.$$

A computation shows that

$$\psi \circ \mathcal{G}(\beta_1(s_1), s_2) = \beta H(C, s_2) + (1 - \beta)H(D, s_2)$$

and

$$\psi \circ \mathcal{F}(s_1, s_2) = \beta H(A, s_2) + (1 - \beta)H(B, s_2).$$

On account of (2.4), we have

$$\psi \circ \mathcal{G}(\beta_1(s_1), s_2) \leq \psi \circ \mathcal{F}(s_1, s_2),$$

therefore

$$\psi \circ \mathcal{G}(\beta_1(s_1), \beta_2(s_2)) \leq \psi \circ \mathcal{G}(\beta_1(s_1), s_2) \leq \psi \circ \mathcal{F}(s_1, s_2)$$

which establishes the desired inequality.

3. We proceed similarly as in the proof of part 2, with β and $(0, s_1] \times (0, s_2]$, respectively, replaced by γ and $[s_1, 1) \times [s_2, 1)$, we can assert that $\mathcal{F} \circ \gamma$, $\mathcal{J}$ and $\mathcal{G} \circ \gamma$ increase co-ordinates on $[0, 1)^2$ and (2.3) follows. If $w_i = \theta_i, i = 1, 2$, then $\mathcal{I}(1, 1) = \mathcal{J}(0, 0)$, which is clear from the definitions of $\mathcal{I}$ and $\mathcal{J}$. $\square$

Let us now mention another important consequence of Theorem 2.1. It should be pointed out that a variety of Fejér type inequalities for co-ordinates $(\mathcal{M}_\phi, \mathcal{M}_\psi)$ -convex functions can be produced by choosing various weights, w_1, w_2 and θ_1, θ_2 . For instance, let us choose

$$w_i(t_i) = (1 - \alpha)g_i(t_i) \circ \mathcal{L}_i(t_i) + \alpha g_i(t_i) \circ \mathcal{R}_i(t_i), \quad t_i \in [0, 1], i = 1, 2$$

and

$$\theta_i(t_i) = (1 - \alpha)h_i(t_i) \circ \mathcal{L}_i(t_i) + \alpha h_i(t_i) \circ \mathcal{R}_i(t_i), \quad t_i \in [0, 1], i = 1, 2$$

where $g_i, h_i : I \rightarrow [0, \infty)$ are given to satisfy

$$\frac{1 - \alpha}{\alpha} g_i(t_i) \circ \mathcal{L}_i(t_i) = \frac{\alpha}{1 - \alpha} g_i(t_i) \circ \mathcal{R}_i(t_i), \quad t_i \in [0, s_i] \quad (2.11)$$

and

$$\frac{1 - \alpha}{\alpha} h_i(t_i) \circ \mathcal{L}_i(t_i) = \frac{\alpha}{1 - \alpha} h_i(t_i) \circ \mathcal{R}_i(t_i), \quad t_i \in [s_i, 1]. \quad (2.12)$$

Notice that if $\alpha = 1/2$ and $\phi(x) = x$, then the assumptions (2.11) and (2.12) reduce to the ones that g_i and h_i are symmetric to $(a_i + b_i)/2$ for $i = 1, 2$.

A computation, using (2.11) and $\mathcal{L}(0,0) = \mathcal{R}(0,0)$, forces

$$\begin{aligned}
& \int_0^{s_1} w_1(t_1) dt_1 \int_0^{s_2} w_2(t_2) dt_2 \\
&= \left((1-\alpha) \int_0^{s_1} g_1(t_1) \circ \mathcal{L}_1(t_1) dt_1 + \alpha \int_0^{s_1} g_1(t_1) \circ \mathcal{R}_1(t_1) dt_1 \right) \\
&\quad \times \left((1-\alpha) \int_0^{s_2} g_2(t_2) \circ \mathcal{L}_2(t_2) dt_2 + \alpha \int_0^{s_2} g_2(t_2) \circ \mathcal{R}_2(t_2) dt_2 \right) \\
&= \frac{1}{(\phi(b_1) - \phi(a_1))(\phi(b_2) - \phi(a_2))} \int_{\mathcal{L}_1(s_1)}^{\mathcal{R}_1(s_1)} g_1(x) d\phi(x) \int_{\mathcal{L}_2(s_2)}^{\mathcal{R}_2(s_2)} g_2(x) d\phi(x),
\end{aligned}$$

and

$$\begin{aligned}
& \int_0^{s_1} \int_0^{s_2} \psi \circ \mathcal{F}(t_1, t_2) w_1(t_1) w_2(t_2) dt_1 dt_2 \\
&= \alpha \int_0^{s_1} \left(\int_0^{s_2} [\alpha \psi \circ f(\mathcal{L}_1(t_1), \mathcal{L}_2(t_2)) + (1-\alpha) \psi \circ f(\mathcal{L}_1(t_1), \mathcal{R}_2(t_2))] w_2(t_2) dt_2 \right) w_1(t_1) dt_1 \\
&\quad + (1-\alpha) \int_0^{s_1} \left(\int_0^{s_2} [\alpha \psi \circ f(\mathcal{R}_1(t_1), \mathcal{L}_2(t_2)) + (1-\alpha) \psi \circ f(\mathcal{R}_1(t_1), \mathcal{R}_2(t_2))] w_2(t_2) dt_2 \right) w_1(t_1) dt_1 \\
&= \frac{\alpha}{\phi(b_2) - \phi(a_2)} \int_0^{s_1} \left(\int_{\mathcal{L}_2(s_2)}^{\mathcal{R}_2(s_2)} \psi \circ f(\mathcal{L}_1(t_1), x_2) g_2(x_2) d\phi(x_2) \right) w_1(t_1) dt_1 \\
&\quad + \frac{1-\alpha}{\phi(b_2) - \phi(a_2)} \int_0^{s_1} \left(\int_{\mathcal{L}_2(s_2)}^{\mathcal{R}_2(s_2)} \psi \circ f(\mathcal{R}_1(t_1), x_2) g_2(x_2) d\phi(x_2) \right) w_1(t_1) dt_1 \\
&= \frac{1}{\phi(b_2) - \phi(a_2)} \int_{\mathcal{L}_2(s_2)}^{\mathcal{R}_2(s_2)} \left(\int_0^{s_1} [\alpha \psi \circ f(\mathcal{L}_1(t_1), x_2) + (1-\alpha) \psi \circ f(\mathcal{R}_1(t_1), x_2)] w_1(t_1) dt_1 \right) g_2(x_2) d\phi(x_2) \\
&= \frac{1}{(\phi(b_1) - \phi(a_1))(\phi(b_2) - \phi(a_2))} \int_{\mathcal{L}_1(s_1)}^{\mathcal{R}_1(s_1)} \int_{\mathcal{L}_2(s_2)}^{\mathcal{R}_2(s_2)} \psi \circ f(x_1, x_2) g_1(x_1) g_2(x_2) d\phi(x_1) d\phi(x_2)
\end{aligned}$$

and hence

$$\mathcal{I}(s_1, s_2) = \psi^{-1} \left(\frac{\int_{\mathcal{L}_1(s_1)}^{\mathcal{R}_1(s_1)} \int_{\mathcal{L}_2(s_2)}^{\mathcal{R}_2(s_2)} \psi \circ f(x_1, x_2) g_1(x_1) g_2(x_2) d\phi(x_1) d\phi(x_2)}{\int_{\mathcal{L}_1(s_1)}^{\mathcal{R}_1(s_1)} g_1(x_1) d\phi(x_1) \int_{\mathcal{L}_2(s_2)}^{\mathcal{R}_2(s_2)} g_2(x_2) d\phi(x_2)} \right).$$

Similarly, by (2.12), $\mathcal{L}(1,1) = (a_1, a_2)$ and $\mathcal{R}(1,1) = (b_1, b_2)$,

$$\mathcal{J}(s_1, s_2) = \psi^{-1} \left(\frac{\mathcal{J}_1(s_1, s_2)}{\mathcal{J}_2(s_1, s_2)} \right),$$

where

$$\begin{aligned}
\mathcal{J}_1(s_1, s_2) = & \int_{a_1}^{\mathcal{L}_1(s_1)} \int_{a_2}^{\mathcal{L}_2(s_2)} \psi \circ f(x_1, x_2) h_1(x_1) h_2(x_2) d\phi(x_1) d\phi(x_2) \\
& + \int_{a_1}^{\mathcal{L}_1(s_1)} \int_{\mathcal{R}_2(s_2)}^{b_2} \psi \circ f(x_1, x_2) h_1(x_1) h_2(x_2) d\phi(x_1) d\phi(x_2) \\
& + \int_{\mathcal{R}_1(s_1)}^{b_1} \int_{a_2}^{\mathcal{L}_2(s_2)} \psi \circ f(x_1, x_2) h_1(x_1) h_2(x_2) d\phi(x_1) d\phi(x_2) \\
& + \int_{\mathcal{R}_1(s_1)}^{b_1} \int_{\mathcal{R}_2(s_2)}^{b_2} \psi \circ f(x_1, x_2) h_1(x_1) h_2(x_2) d\phi(x_1) d\phi(x_2)
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{J}_2(s_1, s_2) = & \left(\int_{a_1}^{\mathcal{L}_1(s_1)} h_1(x_1) d\phi(x_1) + \int_{\mathcal{R}_1(s_1)}^b h_1(x_1) d\phi(x_1) \right) \\
& \times \left(\int_{a_2}^{\mathcal{L}_2(s_2)} h_2(x_2) d\phi(x_2) + \int_{\mathcal{R}_2(s_2)}^b h_2(x_2) d\phi(x_2) \right).
\end{aligned}$$

Together with the aid of Theorem 2.1, we establish the following corollary.

Corollary 2.6. *Suppose that $g_i, h_i : [a_i, b_i] \rightarrow [0, \infty)$ are integrable for $i = 1, 2$, with $\int_0^{s_i} g_i \circ \mathcal{L}_i(t_i) dt_i > 0$ and $\int_{s_i}^1 h_i \circ \mathcal{R}_i(t_i) dt_i > 0$ for all $s_i \in (0, 1)$, and satisfy (2.11) and (2.12). Then, for $s = (s_1, s_2) \in (0, 1)^2$,*

$$\begin{aligned}
f(\mathcal{M}_\phi(a, b; \alpha)) & \leq \mathcal{F} \left(\frac{\int_0^{s_1} t_1 g_1 \circ \mathcal{L}_1(t_1) dt_1}{\int_0^{s_1} g_1 \circ \mathcal{L}_1(t_1) dt_1}, \frac{\int_0^{s_2} t_2 g_2 \circ \mathcal{L}_2(t_2) dt_2}{\int_0^{s_2} g_2 \circ \mathcal{L}_2(t_2) dt_2} \right) \\
& \leq \psi^{-1} \left(\frac{\int_{\mathcal{L}_1(s_1)}^{\mathcal{R}_1(s_1)} \int_{\mathcal{L}_2(s_2)}^{\mathcal{R}_2(s_2)} \psi \circ f(x_1, x_2) g_1(x_1) g_2(x_2) d\phi(x_1) d\phi(x_2)}{\int_{\mathcal{L}_1(s_1)}^{\mathcal{R}_1(s_1)} g_1(x_1) d\phi(x_1) \int_{\mathcal{L}_2(s_2)}^{\mathcal{R}_2(s_2)} g_2(x_2) d\phi(x_2)} \right) \\
& \leq \mathcal{G} \left(\frac{\int_0^{s_1} t_1 g_1 \circ \mathcal{L}_1(t_1) dt_1}{\int_0^{s_1} g_1 \circ \mathcal{L}_1(t_1) dt_1}, \frac{\int_0^{s_2} t_2 g_2 \circ \mathcal{L}_2(t_2) dt_2}{\int_0^{s_2} g_2 \circ \mathcal{L}_2(t_2) dt_2} \right) \\
& \leq \mathcal{F}(s) \leq \mathcal{H}(s) \leq \mathcal{G}(s) \tag{2.13} \\
& \leq \mathcal{F} \left(\frac{\int_{s_1}^1 t_1 h_1 \circ \mathcal{L}_1(t_1) dt_1}{\int_{s_1}^1 h_1 \circ \mathcal{L}_1(t_1) dt_1}, \frac{\int_{s_2}^1 t_2 h_2 \circ \mathcal{L}_2(t_2) dt_2}{\int_{s_2}^1 h_2 \circ \mathcal{L}_2(t_2) dt_2} \right) \\
& \leq \psi^{-1} \left(\frac{\mathcal{J}_1(s_1, s_2)}{\mathcal{J}_2(s_1, s_2)} \right) \\
& \leq \mathcal{G} \left(\frac{\int_{s_1}^1 t_1 h_1 \circ \mathcal{L}_1(t_1) dt_1}{\int_{s_1}^1 h_1 \circ \mathcal{L}_1(t_1) dt_1}, \frac{\int_{s_2}^1 t_2 h_2 \circ \mathcal{L}_2(t_2) dt_2}{\int_{s_2}^1 h_2 \circ \mathcal{L}_2(t_2) dt_2} \right) \leq \mathcal{G}(1, 1).
\end{aligned}$$

Remark 2.7. It turns out that a great deal of existing inequalities of Hermite-Hadamard and Fejér type for different kinds of convexity can be attained from Corollary 2.6.

1. Let us consider $\phi(x) = x$ and $\psi(x) = x$. If $\alpha = 1/2$ then (2.13) offers a refinement of the inequalities due to Farid, Marwan and Atiq Ur Rehman [15].
2. Next, if we choose $\phi(x) = x$ and $\psi(x) = x$. If $\alpha = 1/2$, $g_1 = g_2 = h_1 = h_2 = 1$ and $s_1 = s_2 = 1/2$, (2.13) implies

$$\begin{aligned}
& f\left(\frac{a_1 + b_1}{2}, \frac{a_2 + b_2}{2}\right) \\
& \leq \frac{1}{4} \left[f\left(\frac{5a_1 + 3b_1}{8}, \frac{5a_2 + 3b_2}{8}\right) + f\left(\frac{5a_1 + 3b_1}{8}, \frac{3a_2 + 5b_2}{8}\right) \right. \\
& \quad \left. + f\left(\frac{3a_1 + 5b_1}{8}, \frac{5a_2 + 3b_2}{8}\right) + f\left(\frac{3a_1 + 5b_1}{8}, \frac{3a_2 + 5b_2}{8}\right) \right] \\
& \leq \frac{4}{(b_1 - a_1)(b_2 - a_2)} \int_{(3a_1+b_1)/4}^{(a_1+3b_1)/4} \int_{(3a_2+b_2)/4}^{(a_2+3b_2)/4} f(x_1, x_2) dx_1 dx_2 \\
& \leq \frac{1}{16} \left[\frac{f(a_1, a_2) + f(a_1, b_2) + f(b_1, a_2) + f(b_1, b_2)}{4} + 3 \frac{f(a_1, \frac{a_2+b_2}{2}) + f(b_1, \frac{a_2+b_2}{2})}{2} \right. \\
& \quad \left. + 3 \frac{f(\frac{a_1+b_1}{2}, a_2) + f(\frac{a_1+b_1}{2}, b_2)}{2} + f\left(\frac{a_1 + b_1}{2}, \frac{a_2 + b_2}{2}\right) \right] \\
& \leq \frac{1}{4} \left[f\left(\frac{3a_1 + b_1}{4}, \frac{3a_2 + b_2}{4}\right) + f\left(\frac{3a_1 + b_1}{4}, \frac{a_2 + 3b_2}{4}\right) \right. \\
& \quad \left. + f\left(\frac{a_1 + 3b_1}{4}, \frac{3a_2 + b_2}{4}\right) + f\left(\frac{a_1 + 3b_1}{4}, \frac{a_2 + 3b_2}{4}\right) \right] \\
& \leq \frac{1}{(b_1 - a_1)(b_2 - a_2)} \int_{a_1}^{b_1} \int_{a_2}^{b_2} f(x_1, x_2) dx_1 dx_2 \\
& \leq \frac{1}{8} \left[\frac{1}{b_1 - a_1} \int_{a_1}^{b_1} \left(f(x_1, a_2) + f(x_1, b_2) + 2f\left(x_1, \frac{a_2 + b_2}{2}\right) \right) dx_1 \right. \\
& \quad \left. + \frac{1}{b_2 - a_2} \int_{a_2}^{b_2} \left(f(a_1, x_2) + f(b_1, x_2) + 2f\left(\frac{a_1 + b_1}{2}, x_2\right) \right) dx_2 \right] \\
& \leq \frac{1}{4} \left[\frac{f(a_1, a_2) + f(a_1, b_2) + f(b_1, a_2) + f(b_1, b_2)}{4} + \frac{f(a_1, \frac{a_2+b_2}{2}) + f(b_1, \frac{a_2+b_2}{2})}{2} \right. \\
& \quad \left. + \frac{f(\frac{a_1+b_1}{2}, a_2) + f(\frac{a_1+b_1}{2}, b_2)}{2} + f\left(\frac{a_1 + b_1}{2}, \frac{a_2 + b_2}{2}\right) \right] \\
& \leq \frac{f(a_1, a_2) + f(a_1, b_2) + f(b_1, a_2) + f(b_1, b_2)}{4}
\end{aligned} \tag{2.14}$$

which offers a refinement of (2.13) and ones due to Bakula [3], Özdemir [37].

3. Moreover, if $f(x_1, x_2) = f(x_1)g(x_2)$ then (2.13) implies

$$\begin{aligned}
& f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right) \\
& \leq \frac{1}{4} \left[f\left(\frac{5a+3b}{8}\right)g\left(\frac{5a+3b}{8}\right) + f\left(\frac{5a+3b}{8}\right)g\left(\frac{3a+5b}{8}\right) \right. \\
& \quad \left. + f\left(\frac{3a+5b}{8}\right)g\left(\frac{5a+3b}{8}\right) + f\left(\frac{3a+5b}{8}\right)g\left(\frac{3a+5b}{8}\right) \right] \\
& \leq \frac{2}{(b-a)} \int_{(3a+b)/4}^{(a+3b)/4} f(x)g(x)dx \\
& \leq \frac{1}{16} \left[\frac{f(a)g(a) + f(a)g(b) + f(b)g(a) + f(b)g(b)}{4} + 3 \frac{f(a)g(\frac{a+b}{2}) + f(b)g(\frac{a+b}{2})}{2} \right. \\
& \quad \left. + 3 \frac{f(\frac{a+b}{2})g(a) + f(\frac{a+b}{2})g(b)}{2} + f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right) \right] \\
& \leq \frac{1}{4} \left[f\left(\frac{3a+b}{4}\right)g\left(\frac{3a+b}{4}\right) + f\left(\frac{3a+b}{4}\right)g\left(\frac{a+3b}{4}\right) \right. \\
& \quad \left. + f\left(\frac{a+3b}{4}\right)g\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right)g\left(\frac{a+3b}{4}\right) \right] \\
& \leq \frac{1}{(b-a)} \int_a^b f(x)g(x)dx \\
& \leq \frac{1}{8(b-a)} \left[\left(g(a) + g(b) + 2g\left(\frac{a+b}{2}\right) \right) \int_a^b f(x)dx \right. \\
& \quad \left. + \left(f(a) + f(b) + 2f\left(\frac{a+b}{2}\right) \right) \int_a^b g(x)dx \right] \\
& \leq \frac{1}{4} \left[\frac{f(a)g(a) + f(a)g(b) + f(b)g(a) + f(b)g(b)}{4} + \frac{f(a)g(\frac{a+b}{2}) + f(b)g(\frac{a+b}{2})}{2} \right. \\
& \quad \left. + \frac{f(\frac{a+b}{2})g(a) + f(\frac{a+b}{2})g(b)}{2} + f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right) \right] \\
& \leq \frac{f(a)g(a) + f(a)g(b) + f(b)g(a) + f(b)g(b)}{4}.
\end{aligned} \tag{2.15}$$

Let us consider kernels, say $K_i : \phi_i(I) \times \phi_i(I) \rightarrow [0, \infty)$, for $i = 1, 2$ and define

$$\mathcal{K}_{a_1+, a_2+}^{\phi_1, \phi_2}[f](x_1, x_2) = \int_{a_1}^{x_1} \int_{a_2}^{x_2} K_1(\phi_1(x_1), \phi_1(y_1)) K_2(\phi_2(x_2), \phi_2(y_2)) f(y_1, y_2) d\phi(y_1) d\phi(y_2) \tag{2.16}$$

for $x_1 > a_1, x_2 > a_2$,

$$\mathcal{K}_{a_1+, b_2-}^{\phi_1, \phi_2}[f](x_1, x_2) = \int_{a_1}^{x_1} \int_{x_2}^{b_2} K_1(\phi_1(x_1), \phi_1(y_1)) K_2(\phi_2(x_2), \phi_2(y_2)) f(y_1, y_2) d\phi(y_1) d\phi(y_2) \quad (2.17)$$

for $x_1 > a_1, x_2 < b_2$,

$$\mathcal{K}_{b_1-, a_2+}^{\phi_1, \phi_2}[f](x_1, x_2) = \int_{x_1}^{b_1} \int_{a_2}^{x_2} K_1(\phi_1(x_1), \phi_1(y_1)) K_2(\phi_2(x_2), \phi_2(y_2)) f(y_1, y_2) d\phi(y_1) d\phi(y_2) \quad (2.18)$$

for $x_1 < b_1, x_2 > a_2$,

$$\mathcal{K}_{b_1-, b_2-}^{\phi_1, \phi_2}[f](x_1, x_2) = \int_{x_1}^{b_1} \int_{x_2}^{b_2} K_1(\phi_1(x_1), \phi_1(y_1)) K_2(\phi_2(x_2), \phi_2(y_2)) f(y_1, y_2) d\phi(y_1) d\phi(y_2) \quad (2.19)$$

for $x_1 < b_1, x_2 < b_2$, as long as the integrals exist and are finite.

Remark 2.8. We emphasize that our definition agrees with many known fractional integrals existing in the literature as special cases.

1. Let us first consider

$$K_i(u, v) = \frac{1}{\Gamma(\nu_i)} |u - v|^{\nu_i - 1}, \quad u, v \in \phi_i(I), i = 1, 2,$$

where $\nu_i > 0$. Then integral operators (2.16)–(2.19) become the fractional integrals of a function with respect to another function known:

$$\begin{aligned} & \mathcal{I}_{a_1+, a_2+}^{\phi_1, \phi_2}[f](x_1, x_2) \\ &= \frac{1}{\Gamma(\nu_1)\Gamma(\nu_2)} \int_{a_1}^{x_1} \int_{a_2}^{x_2} |\phi_1(x_1) - \phi_1(y_1)|^{\nu_1 - 1} |\phi_2(x_2) - \phi_2(y_2)|^{\nu_2 - 1} f(y_1, y_2) d\phi(y_1) d\phi(y_2) \end{aligned}$$

for $x_1 > a_1, x_2 > a_2$,

$$\begin{aligned} & \mathcal{I}_{a_1+, b_2-}^{\phi_1, \phi_2}[f](x_1, x_2) \\ &= \frac{1}{\Gamma(\nu_1)\Gamma(\nu_2)} \int_{a_1}^{x_1} \int_{x_2}^{b_2} |\phi_1(x_1) - \phi_1(y_1)|^{\nu_1 - 1} |\phi_2(x_2) - \phi_2(y_2)|^{\nu_2 - 1} f(y_1, y_2) d\phi(y_1) d\phi(y_2) \end{aligned}$$

for $x_1 > a_1, x_2 < b_2$,

$$\begin{aligned} & \mathcal{I}_{b_1-, a_2+}^{\phi_1, \phi_2}[f](x_1, x_2) \\ &= \frac{1}{\Gamma(\nu_1)\Gamma(\nu_2)} \int_{x_1}^{b_1} \int_{a_2}^{x_2} |\phi_1(x_1) - \phi_1(y_1)|^{\nu_1 - 1} |\phi_2(x_2) - \phi_2(y_2)|^{\nu_2 - 1} f(y_1, y_2) d\phi(y_1) d\phi(y_2) \end{aligned}$$

for $x_1 < b_1, x_2 > a_2$,

$$\begin{aligned} & \mathcal{I}_{b_1-, b_2-}^{\phi_1, \phi_2}[f](x_1, x_2) \\ &= \frac{1}{\Gamma(\nu_1)\Gamma(\nu_2)} \int_{x_1}^{b_1} \int_{x_2}^{b_2} |\phi_1(x_1) - \phi_1(y_1)|^{\nu_1 - 1} |\phi_2(x_2) - \phi_2(y_2)|^{\nu_2 - 1} f(y_1, y_2) d\phi(y_1) d\phi(y_2) \end{aligned}$$

for $x_1 < b_1, x_2 < b_2$,

These operators include the Riemann-Liouville fractional integral [41], which the choice $\phi(x) = x$.

In Corollary 2.6, for $s_i \in (0, 1), i = 1, 2$, let us choose

$$g_i(x) = [K_i(\phi_i \circ \mathcal{R}_i(s_i), \phi_i(x_i)) + K_i(\phi_i \circ \mathcal{L}_i(s_i), \phi_i(x_i))] u_i(x_i), \quad x_i \in [\mathcal{L}_i(s_i), \mathcal{R}_i(s_i)], i = 1, 2$$

and

$$h_i(x_i) = \begin{cases} K_i(\phi_i \circ \mathcal{L}_i(s_i), \phi_i(x_i))v_i(x_i) & \text{if } x_i \in [a_i, \mathcal{L}_i(s_i)], \\ K_i(\phi_i \circ \mathcal{R}_i(s_i), \phi_i(x_i))v_i(x_i) & \text{if } x_i \in [\mathcal{R}_i(s_i), b_i], \end{cases}$$

where $u_i, v_i : [a_i, b_i] \rightarrow [0, \infty)$, for $i = 1, 2$, are given in such a way that the assumptions (2.11) and (2.12) are guaranteed, i.e.,

$$\begin{aligned} & \frac{1-\alpha}{\alpha} [K_i(\phi_i \circ \mathcal{R}_i(s_i), \phi_i \circ \mathcal{L}_i(t_i)) + K_i(\phi_i \circ \mathcal{L}_i(s_i), \phi_i \circ \mathcal{L}_i(t_i))] u_i \circ \mathcal{L}_i(t_i) \\ &= \frac{\alpha}{1-\alpha} [K_i(\phi_i \circ \mathcal{R}_i(s_i), \phi_i \circ \mathcal{R}_i(t_i)) + K_i(\phi_i \circ \mathcal{L}_i(s_i), \phi_i \circ \mathcal{R}_i(t_i))] u_i \circ \mathcal{R}_i(t_i), \end{aligned} \quad (2.20)$$

for $t_i \in [0, s_i], i = 1, 2$ and

$$\frac{1-\alpha}{\alpha} K_i(\phi_i \circ \mathcal{L}_i(s_i), \phi_i \circ \mathcal{L}_i(t_i))v_i \circ \mathcal{L}_i(t_i) = \frac{\alpha}{1-\alpha} K_i(\phi_i \circ \mathcal{R}_i(s_i), \phi_i \circ \mathcal{R}_i(t_i))v_i \circ \mathcal{R}_i(t_i), \quad (2.21)$$

for $t_i \in [s_i, 1], i = 1, 2$.

In order to simplify these assumptions, it is necessary to put some restrictions on α and $K_i, i = 1, 2$. Let us take $\alpha = 1/2$ and investigate a class of kernels, K_i , of the form

$$K_i(u, v) = k_i(|u - v|), \quad u, v \in \phi_i(I), i = 1, 2, \quad (2.22)$$

where $k_i : [0, \infty) \rightarrow [0, \infty)$ for $i = 1, 2$ are given so that the integral operators (2.16)-(2.19) are well-defined.

We check at once that

$$\begin{aligned} |\phi_i \circ \mathcal{L}_i(s_i) - \phi_i \circ \mathcal{L}_i(t_i)| &= |\phi_i \circ \mathcal{R}_i(s_i) - \phi_i \circ \mathcal{R}_i(t_i)| = \frac{1}{2}|s_i - t_i||\phi(b_i) - \phi(a_i)|, i = 1, 2 \\ |\phi_i \circ \mathcal{L}_i(s_i) - \phi_i \circ \mathcal{R}_i(t_i)| &= |\phi_i \circ \mathcal{R}_i(s_i) - \phi_i \circ \mathcal{L}_i(t_i)| = \frac{1}{2}(s_i + t_i)|\phi(b_i) - \phi(a_i)|, i = 1, 2. \end{aligned}$$

Consequently, (2.20) and (2.21) reduce to

$$u_i \circ \mathcal{L}_i(t_i) = u_i \circ \mathcal{R}_i(t_i), \quad t_i \in [0, s_i], i = 1, 2$$

and

$$v_i \circ \mathcal{L}_i(t_i) = v_i \circ \mathcal{R}_i(t_i), \quad t_i \in [s_i, 1], i = 1, 2$$

respectively. This enables one to take

$$m_i(x_i) = \begin{cases} u_i(x_i) & \text{if } x_i \in [\mathcal{L}_i(s_i), \mathcal{R}_i(s_i)], \\ v_i(x_i) & \text{otherwise,} \end{cases}$$

for $i = 1, 2$. Put this way, we have

$$\begin{aligned} \beta_i(s_i) &= \frac{\mathcal{K}_{\mathcal{L}_i(s_i)+}^{\phi_i}[\varphi_i m_i](\mathcal{R}_i(s_i)) + \mathcal{K}_{\mathcal{R}_i(s_i)-}^{\phi_i}[\varphi_i m_i](\mathcal{L}_i(s_i))}{\mathcal{K}_{\mathcal{L}_i(s_i)+}^{\phi_i}[m_i](\mathcal{R}_i(s_i)) + \mathcal{K}_{\mathcal{R}_i(s_i)-}^{\phi_i}[m_i](\mathcal{L}_i(s_i))}, \\ \gamma_i(s_i) &= \frac{\mathcal{K}_{a+}^{\phi_i}[\varphi_i m_i](\mathcal{L}_i(s_i)) + \mathcal{K}_{b-}^{\phi_i}[\varphi_i m_i](\mathcal{R}_i(s_i))}{\mathcal{K}_{a+}^{\phi_i}[m_i](\mathcal{L}_i(s_i)) + \mathcal{K}_{b-}^{\phi_i}[m_i](\mathcal{R}_i(s_i))}, \end{aligned}$$

where

$$\varphi_i(x_i) = \left| \frac{\phi_i(a_i) + \phi_i(b_i) - 2\phi_i(x_i)}{\phi_i(b_i) - \phi_i(a_i)} \right|, \quad x_i \in I, i = 1, 2. \quad (2.23)$$

Set

$$\begin{aligned} \mathcal{K}^1(s) = \mathcal{C}^1(s) & \left(\mathcal{K}_{\mathcal{L}_1(s_1)+, \mathcal{L}_2(s_2)+}^{\phi_1, \phi_2}[(\psi \circ f)m_1 m_2](\mathcal{R}_1(s_1), \mathcal{R}_2(s_2)) \right. \\ & + \mathcal{K}_{\mathcal{L}_1(s_1)+, \mathcal{R}_2(s_2)-}^{\phi_1, \phi_2}[(\psi \circ f)m_1 m_2](\mathcal{R}_1(s_1), \mathcal{L}_2(s_2)) \\ & + \mathcal{K}_{\mathcal{R}_1(s_1)-, \mathcal{L}_2(s_2)+}^{\phi_1, \phi_2}[(\psi \circ f)m_1 m_2](\mathcal{L}_1(s_1), \mathcal{R}_2(s_2)) \\ & \left. + \mathcal{K}_{\mathcal{R}_1(s_1)-, \mathcal{R}_2(s_2)-}^{\phi_1, \phi_2}[(\psi \circ f)m_1 m_2](\mathcal{L}_1(s_1), \mathcal{L}_2(s_2)) \right) \end{aligned}$$

where

$$\begin{aligned} \mathcal{C}^1(s) &= \frac{1}{\mathcal{K}_{\mathcal{L}_1(s_1)+}^{\phi_1}[m_1](\mathcal{R}_1(s_1)) + \mathcal{K}_{\mathcal{R}_1(s_1)-}^{\phi_1}[m_1](\mathcal{L}_1(s_1))} \\ & \times \frac{1}{\mathcal{K}_{\mathcal{L}_2(s_2)+}^{\phi_2}[m_2](\mathcal{R}_2(s_2)) + \mathcal{K}_{\mathcal{R}_2(s_2)-}^{\phi_2}[m_2](\mathcal{L}_2(s_2))} \end{aligned}$$

and

$$\begin{aligned} \mathcal{K}^2(s) = \mathcal{C}^1(s) & \left(\mathcal{K}_{a_1+, a_2+}^{\phi_1, \phi_2}[(\psi \circ f)m_1 m_2](\mathcal{L}_1(s_1), \mathcal{L}_2(s_2)) \right. \\ & + \mathcal{K}_{a_1+, b_2-}^{\phi_1, \phi_2}[(\psi \circ f)m_1 m_2](\mathcal{L}_1(s_1), \mathcal{R}_2(s_2)) \\ & + \mathcal{K}_{b_1-, a_2+}^{\phi_1, \phi_2}[(\psi \circ f)m_1 m_2](\mathcal{R}_1(s_1), \mathcal{L}_2(s_2)) \\ & \left. + \mathcal{K}_{b_1-, b_2-}^{\phi_1, \phi_2}[(\psi \circ f)m_1 m_2](\mathcal{R}_1(s_1), \mathcal{R}_2(s_2)) \right) \end{aligned}$$

where

$$\begin{aligned}\mathcal{C}^2(s) &= \frac{1}{\mathcal{K}_{a_1+}^{\phi}[m_1](\mathcal{L}_1(s_1)) + \mathcal{K}_{b_1-}^{\phi}[m_1](\mathcal{R}_1(s_1))} \\ &\quad \times \frac{1}{\mathcal{K}_{a_2+}^{\phi}[m_2](\mathcal{L}_2(s_2)) + \mathcal{K}_{b_2-}^{\phi}[m_2](\mathcal{R}_2(s_2))}\end{aligned}$$

In summary, we get the following corollary.

Corollary 2.9. *Let φ be given by (2.23) and $\alpha = 1/2$. Suppose that $K_i : \phi_i(I) \times \phi_i(I) \rightarrow [0, \infty)$ is of the form (2.22) and $m_i : [a_i, b_i] \rightarrow [0, \infty)$ is integrable such that*

$$\frac{\mathcal{K}_{\mathcal{L}_i(s_i)+}^{\phi_i}[m_i](\mathcal{R}_i(s_i)) + \mathcal{K}_{\mathcal{R}_i(s_i)-}^{\phi_i}[m_i](\mathcal{L}_i(s_i))}{\phi_i(b_i) - \phi_i(a_i)} > 0 \quad \text{and} \quad \frac{\mathcal{K}_{a+}^{\phi_i}[m_i](\mathcal{L}_i(s_i)) + \mathcal{K}_{b-}^{\phi_i}[m_i](\mathcal{R}_i(s_i))}{\phi_i(b_i) - \phi_i(a_i)} > 0$$

for all $s_i \in (0, 1)$. If

$$m_i \circ \mathcal{L}_i(t_i) = m_i \circ \mathcal{R}_i(t_i), \quad t_i \in [0, 1], \quad (2.24)$$

then

$$\begin{aligned}f(\mathcal{M}_{\phi_1}(a_1, b_2), \mathcal{M}_{\phi_2}(a_2, b_2)) &\leq \mathcal{F}(\beta_1(s_1), \beta_2(s_2)) \\ &\leq \psi^{-1}(\mathcal{K}^1(s)) \\ &\leq \mathcal{G}(\beta_1(s_1), \beta_2(s_2)) \leq \mathcal{F}(s) \\ &\leq \mathcal{G}(s) \leq \mathcal{F}(\gamma_1(s_1), \gamma_2(s_2)) \\ &\leq \psi^{-1}(\mathcal{K}^2(s)) \\ &\leq \mathcal{M}_{\psi}(f(a), f(b)).\end{aligned} \quad (2.25)$$

In particular, one has

$$\begin{aligned}f(\mathcal{M}_{\phi_1}(a_1, b_2), \mathcal{M}_{\phi_2}(a_2, b_2)) &\leq \mathcal{F}\left(\frac{\mathcal{K}_{a_1+}^{\phi_1}[\varphi_1 m_1](b_1) + \mathcal{K}_{b_1-}^{\phi_1}[\varphi_1 m_1](a_1)}{\mathcal{K}_{a_1+}^{\phi_1}[m_1](b_1) + \mathcal{K}_{b_1-}^{\phi_1}[m_1](a_1)}, \frac{\mathcal{K}_{a_2+}^{\phi_2}[\varphi_2 m_2](b_2) + \mathcal{K}_{b_2-}^{\phi_2}[\varphi_2 m_2](a_2)}{\mathcal{K}_{a_2+}^{\phi_2}[m_2](b_2) + \mathcal{K}_{b_2-}^{\phi_2}[m_2](a_2)}\right) \\ &\leq \psi^{-1}\left(\frac{\mathcal{K}_{a_1+, a_2+}^{\phi_1, \phi_2}[(\psi \circ f)m_1 m_2](b_1, b_2) + \mathcal{K}_{a_1+, b_2-}^{\phi_1, \phi_2}[(\psi \circ f)m_1 m_2](b_1, a_2)}{\left[\mathcal{K}_{a_1+}^{\phi_1}[m_1](b_1) + \mathcal{K}_{b_1-}^{\phi_1}[m_1](a_1)\right] \left[\mathcal{K}_{a_2+}^{\phi_2}[m_2](b_2) + \mathcal{K}_{b_2-}^{\phi_2}[m_2](a_2)\right]} \right. \\ &\quad \left. + \frac{\mathcal{K}_{b_1-, a_2+}^{\phi_1, \phi_2}[(\psi \circ f)m_1 m_2](a_1, b_2) + \mathcal{K}_{b_1-, b_2-}^{\phi_1, \phi_2}[(\psi \circ f)m_1 m_2](a_1, a_2)}{\left[\mathcal{K}_{a_1+}^{\phi_1}[m_1](b_1) + \mathcal{K}_{b_1-}^{\phi_1}[m_1](a_1)\right] \left[\mathcal{K}_{a_2+}^{\phi_2}[m_2](b_2) + \mathcal{K}_{b_2-}^{\phi_2}[m_2](a_2)\right]}\right) \\ &\leq \mathcal{G}\left(\frac{\mathcal{K}_{a_1+}^{\phi_1}[\varphi_1 m_1](b_1) + \mathcal{K}_{b_1-}^{\phi_1}[\varphi_1 m_1](a_1)}{\mathcal{K}_{a_1+}^{\phi_1}[m_1](b_1) + \mathcal{K}_{b_1-}^{\phi_1}[m_1](a_1)}, \frac{\mathcal{K}_{a_2+}^{\phi_2}[\varphi_2 m_2](b_2) + \mathcal{K}_{b_2-}^{\phi_2}[\varphi_2 m_2](a_2)}{\mathcal{K}_{a_2+}^{\phi_2}[m_2](b_2) + \mathcal{K}_{b_2-}^{\phi_2}[m_2](a_2)}\right) \\ &\leq \mathcal{M}_{\psi}(f(a), f(b))\end{aligned} \quad (2.26)$$

and

$$\begin{aligned}
& f(\mathcal{M}_1, \mathcal{M}_2) \\
& \leq \mathcal{F} \left(\frac{\mathcal{K}_{a_1+}^{\phi_1}[\varphi_1 m_1](\mathcal{M}_1) + \mathcal{K}_{b_1-}^{\phi_1}[\varphi_1 m_1](\mathcal{M}_1)}{\mathcal{K}_{a_1+}^{\phi_1}[m_1](\mathcal{M}_1) + \mathcal{K}_{b_1-}^{\phi_1}[m_1](\mathcal{M}_1)}, \frac{\mathcal{K}_{a_2+}^{\phi_2}[\varphi_2 m_2](\mathcal{M}_2) + \mathcal{K}_{b_2-}^{\phi_2}[\varphi_2 m_2](\mathcal{M}_2)}{\mathcal{K}_{a_2+}^{\phi_2}[m_2](\mathcal{M}_2) + \mathcal{K}_{b_2-}^{\phi_2}[m_2](\mathcal{M}_2)} \right) \\
& \leq \psi^{-1} \left(\frac{\mathcal{K}_{a_1+,a_2+}^{\phi_1,\phi_2}[(\psi \circ f)m_1 m_2](\mathcal{M}_1, \mathcal{M}_2) + \mathcal{K}_{a_1+,b_2-}^{\phi_1,\phi_2}[(\psi \circ f)m_1 m_2](\mathcal{M}_1, \mathcal{M}_2)}{\left[\mathcal{K}_{a_1+}^{\phi_1}[m_1](\mathcal{M}_1) + \mathcal{K}_{b_1-}^{\phi_1}[m_1](\mathcal{M}_1) \right] \left[\mathcal{K}_{a_2+}^{\phi_2}[m_2](\mathcal{M}_2) + \mathcal{K}_{b_2-}^{\phi_2}[m_2](\mathcal{M}_2) \right]} \right. \\
& \quad \left. + \frac{\mathcal{K}_{b_1-,a_2+}^{\phi_1,\phi_2}[(\psi \circ f)m_1 m_2](\mathcal{M}_1, \mathcal{M}_2) + \mathcal{K}_{b_1-,b_2-}^{\phi_1,\phi_2}[(\psi \circ f)m_1 m_2](\mathcal{M}_1, \mathcal{M}_2)}{\left[\mathcal{K}_{a_1+}^{\phi_1}[m_1](\mathcal{M}_1) + \mathcal{K}_{b_1-}^{\phi_1}[m_1](\mathcal{M}_1) \right] \left[\mathcal{K}_{a_2+}^{\phi_2}[m_2](\mathcal{M}_2) + \mathcal{K}_{b_2-}^{\phi_2}[m_2](\mathcal{M}_2) \right]} \right) \\
& \leq \mathcal{G} \left(\frac{\mathcal{K}_{a_1+}^{\phi_1}[\varphi_1 m_1](\mathcal{M}_1) + \mathcal{K}_{b_1-}^{\phi_1}[\varphi_1 m_1](\mathcal{M}_1)}{\mathcal{K}_{a_1+}^{\phi_1}[m_1](\mathcal{M}_1) + \mathcal{K}_{b_1-}^{\phi_1}[m_1](\mathcal{M}_1)}, \frac{\mathcal{K}_{a_2+}^{\phi_2}[\varphi_2 m_2](\mathcal{M}_2) + \mathcal{K}_{b_2-}^{\phi_2}[\varphi_2 m_2](\mathcal{M}_2)}{\mathcal{K}_{a_2+}^{\phi_2}[m_2](\mathcal{M}_2) + \mathcal{K}_{b_2-}^{\phi_2}[m_2](\mathcal{M}_2)} \right) \\
& \leq \mathcal{M}_{\psi}(f(a), f(b)).
\end{aligned} \tag{2.27}$$

where $\mathcal{M}_1 = \mathcal{M}_{\phi_1}(a_1, b_1)$ and $\mathcal{M}_2 = \mathcal{M}_{\phi_2}(a_2, b_2)$.

Remark 2.10. Through a proper choice of the functions ϕ , ψ and K such as are indicated in Remark 2.8, (2.26) can be regarded as a generalization and refinement of several results obtained recently by Chen [6], Sarıkaya [41].

3. An application to inequalities involving the beta function

We devote this section to establish some inequalities involving the beta function, defined by the integral representation

$$\beta(x, y) = \int_0^1 t^{x-1} (1-t)^{y-1} dt, \quad x > 0, y > 0.$$

It is well-known that the beta function is log-convex on $(0, \infty)^2$ as a function of two variables (see [11]). So it is co-ordinates log-convex on $(0, \infty)^2$.

Fix $a > 0$. Applying Theorem 2.1 for $f(x, y) = \beta(x, y)$, $\phi(x) = x$, $\psi(x) = \ln x$, $a_1 = a_2 = a$, $b_1 = b_2 = a + 1$ and $\alpha = 1/2$ we obtain the following result.

Corollary 3.1. 1. *The functions*

$$\begin{aligned}
\mathcal{F}_3(t_1, t_2) = & \sqrt[4]{\beta \left(a + \frac{1-t_1}{2}, a + \frac{1-t_2}{2} \right) \beta \left(a + \frac{1-t_1}{2}, a + \frac{1+t_2}{2} \right)} \\
& \times \sqrt[4]{\beta \left(a + \frac{1+t_1}{2}, a + \frac{1-t_2}{2} \right) \beta \left(a + \frac{1+t_1}{2}, a + \frac{1+t_2}{2} \right)}
\end{aligned}$$

and

$$\begin{aligned} \mathcal{G}_3(t_1, t_2) &= \left[\beta(a, a) \sqrt[4]{\frac{a^2}{8a(2a+1)}} \right]^{t_1 t_2} \left[\beta\left(a + \frac{1}{2}, a\right) \sqrt{\frac{a}{2a + \frac{1}{2}}} \right]^{t_1(1-t_2) + (1-t_1)t_2} \\ &\quad \times \left[\beta\left(a + \frac{1}{2}, a + \frac{1}{2}\right) \right]^{(1-t_1)(1-t_2)} \end{aligned}$$

are co-ordinates log-convex and co-ordinates increasing on $[0, 1]$, with

$$\beta\left(a + \frac{1}{2}, a + \frac{1}{2}\right) \leq \beta\left(a + \frac{1}{2}, a\right) \sqrt{\frac{a}{2a + \frac{1}{2}}} \leq \mathcal{F}_3(t_1, t_2) \leq \mathcal{G}_3(t_1, t_2) \leq \beta(a, a) \sqrt[4]{\frac{a^2}{8a(2a+1)}}$$

for all $t_1, t_2 \in [0, 1]$.

2. The function

$$\mathcal{P}_3(s_1, s_2) = \exp \left(\frac{1}{s_1 s_2} \int_{a+(1-s_1)/2}^{a+(1+s_1)/2} \int_{a+(1-s_2)/2}^{a+(1+s_2)/2} \ln \beta(x, y) dx dy \right)$$

is co-ordinates increasing on $(0, 1]^2$, with

$$\lim_{s_1 \rightarrow 0^+} \lim_{s_2 \rightarrow 0^+} \mathcal{P}_3(s_1, s_2) = \beta\left(a + \frac{1}{2}, a + \frac{1}{2}\right), \quad \mathcal{P}_3(1, 1) = 2\pi \left(\frac{a}{e}\right)^{2a} \sqrt{\frac{(2a)^{(2a)^2}}{(2a+1)^{(2a+1)^2}}} e^{a+\frac{1}{4}},$$

and

$$\mathcal{F}_3(s_1/2, s_2/2) \leq \mathcal{P}_3(s_1, s_2) \leq \mathcal{G}_3(s_1/2, s_2/2) \leq \mathcal{F}_3(s_1, s_2), \quad s_1, s_2 \in (0, 1].$$

3. The function

$$\begin{aligned} \mathcal{Q}_3(s_1, s_2) &= \exp \left(\frac{1}{(1-s_1)(1-s_2)} \int_a^{a+(1-s_1)/2} \int_a^{a-(1-s_2)/2} \ln \beta(x, y) dx dy \right) \\ &\quad \times \exp \left(\frac{1}{(1-s_1)(1-s_2)} \int_a^{a+(1-s_1)/2} \int_{a+(1-s_2)/2}^{a+1} \ln \beta(x, y) dx dy \right) \\ &\quad \times \exp \left(\frac{1}{(1-s_1)(1-s_2)} \int_{a+(1+s_1)/2}^{a+1} \int_a^{a-(1-s_2)/2} \ln \beta(x, y) dx dy \right) \\ &\quad \times \exp \left(\frac{1}{(1-s_1)(1-s_2)} \int_{a+(1+s_1)/2}^{a+1} \int_{a+(1-s_2)/2}^{a+1} \ln \beta(x, y) dx dy \right) \end{aligned}$$

is co-ordinates increasing on $[0, 1]^2$, with

$$\mathcal{Q}_3(0, 0) = 2\pi \left(\frac{a}{e}\right)^{2a} \sqrt{\frac{(2a)^{(2a)^2}}{(2a+1)^{(2a+1)^2}}} e^{a+\frac{1}{4}}, \quad \lim_{s_1 \rightarrow 1^-} \lim_{s_2 \rightarrow 1^-} \mathcal{Q}_3(s_1, s_2) = \beta(a, a) \sqrt[4]{\frac{a^2}{8a(2a+1)}}$$

and

$$\mathcal{G}_3(s_1, s_2) \leq \mathcal{F}_3\left(\frac{1+s_1}{2}, \frac{1+s_2}{2}\right) \leq \mathcal{Q}_3(s_1, s_2) \leq \mathcal{G}_3\left(\frac{1+s_1}{2}, \frac{1+s_2}{2}\right), \quad s_1, s_2 \in [0, 1].$$

4. In particular,

$$\begin{aligned} \beta\left(a + \frac{1}{2}, a + \frac{1}{2}\right) &\leq \mathcal{F}_3(1/2, 1/2) \\ &\leq 2\pi \left(\frac{a}{e}\right)^{2a} \sqrt{\frac{(2a)^{(2a)^2}}{(2a+1)^{(2a+1)^2}}} e^{a+\frac{1}{4}} \leq \mathcal{G}_3(1/2, 1/2) \leq \beta(a, a) \sqrt[4]{\frac{a^2}{8a(2a+1)}}. \end{aligned}$$

Next, we consider the Mittag-Leffler function, defined by the integral representation

$$E_z(x, y) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(xn + y)}, \quad z \in \mathbb{C}, x, y > 0.$$

The function $\mathbb{E}(x, y) = \Gamma(x)\Gamma(y)E_z(x, y)$ (see Mehrez and Sitnik [18]) is co-ordinates log-convex on $(0, +\infty) \times (0, +\infty)$.

Fix $a > 0$. Applying Theorem 2.1 for $f(x, y) = \mathbb{E}(x, y)$, $\phi(x) = x$, $\psi(x) = \ln x$, $a_1 = a_2 = a$, $b_1 = b_2 = a + 1$ and $\alpha = 1/2$ we obtain the following result.

Corollary 3.2. 1. *The functions*

$$\begin{aligned} \mathcal{F}_3(t_1, t_2) &= \sqrt[4]{\mathbb{E}\left(a + \frac{1-t_1}{2}, a + \frac{1-t_2}{2}\right) \mathbb{E}\left(a + \frac{1-t_1}{2}, a + \frac{1+t_2}{2}\right)} \\ &\quad \times \sqrt[4]{\mathbb{E}\left(a + \frac{1+t_1}{2}, a + \frac{1-t_2}{2}\right) \mathbb{E}\left(a + \frac{1+t_1}{2}, a + \frac{1+t_2}{2}\right)} \end{aligned}$$

and

$$\begin{aligned} \mathcal{G}_3(t_1, t_2) &= \left[a\Gamma^2(a) \sqrt[4]{E_z(a, a)E_z(a, a+1)E_z(a+1, a)E_z(a+1, a+1)} \right]^{t_1 t_2} \\ &\quad \times \left[\sqrt{a}\Gamma\left(a + \frac{1}{2}\right) \Gamma(a) \sqrt{E_z\left(a + \frac{1}{2}, a\right) E_z\left(a + \frac{1}{2}, a+1\right)} \right]^{t_1(1-t_2)} \\ &\quad \times \left[\sqrt{a}\Gamma\left(a + \frac{1}{2}\right) \Gamma(a) \sqrt{E_z\left(a, a + \frac{1}{2}\right) E_z\left(a+1, a + \frac{1}{2}\right)} \right]^{(1-t_1)t_2} \\ &\quad \times \left[\Gamma\left(a + \frac{1}{2}\right) E_z\left(a + \frac{1}{2}, a + \frac{1}{2}\right) \right]^{(1-t_1)(1-t_2)} \end{aligned}$$

are co-ordinates log-convex and co-ordinates increasing on $[0, 1]$, with

$$\begin{aligned} & \Gamma\left(a + \frac{1}{2}\right) E_z\left(a + \frac{1}{2}, a + \frac{1}{2}\right) \\ & \leq \sqrt{a} \Gamma\left(a + \frac{1}{2}\right) \Gamma(a) \sqrt{E_z\left(a, a + \frac{1}{2}\right) E_z\left(a + 1, a + \frac{1}{2}\right)} \\ & \leq \mathcal{F}_3(t_1, t_2) \leq \mathcal{G}_3(t_1, t_2) \\ & \leq a \Gamma^2(a) \sqrt[4]{E_z(a, a) E_z(a, a + 1) E_z(a + 1, a) E_z(a + 1, a + 1)} \end{aligned}$$

for all $t_1, t_2 \in [0, 1]$.

2. The function

$$\mathcal{P}_3(s_1, s_2) = \exp\left(\frac{1}{s_1 s_2} \int_{a+(1-s_1)/2}^{a+(1+s_1)/2} \int_{a+(1-s_2)/2}^{a+(1+s_2)/2} \ln \mathbb{E}(x, y) dx dy\right)$$

is co-ordinates increasing on $(0, 1]^2$, with

$$\begin{aligned} \lim_{s_1 \rightarrow 0^+} \lim_{s_2 \rightarrow 0^+} \mathcal{P}_3(s_1, s_2) &= \Gamma\left(a + \frac{1}{2}\right) E_z\left(a + \frac{1}{2}, a + \frac{1}{2}\right), \\ \mathcal{P}_3(1, 1) &= 2\sqrt{2\pi} \left(\frac{a}{e}\right)^a \exp\left(\int_a^{a+1} \int_a^{a+1} \ln E_z(x, y) dx dy\right), \end{aligned}$$

and

$$\mathcal{F}_3(s_1/2, s_2/2) \leq \mathcal{P}_3(s_1, s_2) \leq \mathcal{G}_3(s_1/2, s_2/2) \leq \mathcal{F}_3(s_1, s_2), \quad s_1, s_2 \in (0, 1].$$

3. The function

$$\begin{aligned} \mathcal{Q}_3(s_1, s_2) &= \exp\left(\frac{1}{(1-s_1)(1-s_2)} \int_a^{a+(1-s_1)/2} \int_a^{a+(1-s_2)/2} \ln \mathbb{E}(x, y) dx dy\right) \\ &\times \exp\left(\frac{1}{(1-s_1)(1-s_2)} \int_a^{a+(1-s_1)/2} \int_{a+(1-s_2)/2}^{a+1} \ln \mathbb{E}(x, y) dx dy\right) \\ &\times \exp\left(\frac{1}{(1-s_1)(1-s_2)} \int_{a+(1+s_1)/2}^{a+1} \int_a^{a+(1-s_2)/2} \ln \mathbb{E}(x, y) dx dy\right) \\ &\times \exp\left(\frac{1}{(1-s_1)(1-s_2)} \int_{a+(1+s_1)/2}^{a+1} \int_{a+(1-s_2)/2}^{a+1} \ln \mathbb{E}(x, y) dx dy\right) \end{aligned}$$

is co-ordinates increasing on $[0, 1)^2$, with

$$\mathcal{Q}_3(0, 0) = 2\sqrt{2\pi} \left(\frac{a}{e}\right)^a \exp\left(\int_a^{a+1} \int_a^{a+1} \ln E_z(x, y) dx dy\right),$$

$$\lim_{s_1 \rightarrow 1^-} \lim_{s_2 \rightarrow 1^-} \mathcal{Q}_3(s_1, s_2) = a\Gamma^2(a) \sqrt[4]{E_z(a, a)E_z(a, a+1)E_z(a+1, a)E_z(a+1, a+1)}$$

and

$$\mathcal{G}_3(s_1, s_2) \leq \mathcal{F}_3\left(\frac{1+s_1}{2}, \frac{1+s_2}{2}\right) \leq \mathcal{Q}_3(s_1, s_2) \leq \mathcal{G}_3\left(\frac{1+s_1}{2}, \frac{1+s_2}{2}\right), \quad s_1, s_2 \in [0, 1].$$

4. In particular,

$$\begin{aligned} \Gamma\left(a + \frac{1}{2}\right) E_z\left(a + \frac{1}{2}, a + \frac{1}{2}\right) &\leq \mathcal{F}_3(1/2, 1/2) \\ &\leq 2\sqrt{2\pi} \left(\frac{a}{e}\right)^a \exp\left(\int_a^{a+1} \int_a^{a+1} \ln E_z(x, y) dx dy\right) \\ &\leq \mathcal{G}_3(1/2, 1/2) \\ &\leq a\Gamma^2(a) \sqrt[4]{E_z(a, a)E_z(a, a+1)E_z(a+1, a)E_z(a+1, a+1)}. \end{aligned}$$

Similar considerations may apply to other special functions provided that these functions are co-ordinates log-convex.

(i) The Struve function, defined by the integral representation

$$M(x, y) = \frac{2}{\sqrt{\pi}} \int_0^1 (1-t^2)^{x-\frac{1}{2}} e^{-yt} dt, \quad x > -\frac{1}{2}, y > 0.$$

is co-ordinates log-convex on $(-\frac{1}{2}, +\infty) \times (0, +\infty)$ (see [26]).

(ii) The Gauss function

$$G(x, y) = {}_2F_1(x, y; c; z) = \sum_{n=0}^{\infty} \frac{(x)_n (y)_n}{(c)_n n!} z^n, \quad c > x > 0, c > y > 0, z < 1.$$

is co-ordinates log-convex on $(0, +\infty) \times (0, +\infty)$ (see [27]).

Remark 3.3. One may develop further inequalities related to the Struve function and Gauss function by applying Theorem 2.1 for $f(x, y) = M(x, y)$ and $f(x, y) = G(x, y)$, $\phi(x) = x$, $\psi(x) = \ln x$.

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