LEVEL PROPERTY OF ORDINARY AND SYMBOLIC POWERS OF STANLEY-REISNER IDEALS

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Abstract. In this paper, we prove that the t-th ordinary and/or symbolic power of a Stanley-Reisner ideal is level for some positive integer $t \geq 3$ if and only if I_Δ is a complete intersection and equi-generated. For $t = 2$, we give a characterzation of level property of the second symbolic power $I_\Delta^{(2)}$ when Δ is a matroid complex of dimension one.

1. Introduction

Let Δ be a simplicial complex on $[n] = \{1, \ldots, n\}$ and $S = K[x_1, \ldots, x_n]$ a polynomial over a field K. The Stanley-Reisner ideal I_Δ of Δ (over K) is the ideal in S which is generated by all square-free monomials $x_{i_1} \ldots x_{i_p}$ such that $\{i_1, \ldots, i_p\} \notin \Delta$. It is known that I_Δ has the primary decomposition $I_\Delta = \bigcap_{F\colon \text{facet of } \Delta} P_F$, where $P_F = (x_i \mid i \in [n] \setminus F)$. Then for $t \geq 1$, the t-th symbolic power $I_\Delta^{(t)}$ of I_Δ is expressed as

$$I_\Delta^{(t)} = \bigcap_{F\colon \text{facet of } \Delta} P_F^t.$$

The purpose of this paper is to study the following question:

Question. When is S/I_Δ^t or $S/I_\Delta^{(t)}$ a level ring for $t \geq 1$?

This question fits into an ongoing research program to characterize ring properties of S/I^t or $S/I^{(t)}$. The Cohen-Macaulayness, the Buchsbaumness, the generalized Cohen-Macaulayness , and the k-Buchsbaumness were studied, for example, in [MT1], [MT2], [TT], [RTY], [HMT], [TY] and [M]. For Cohen-Macaulay case it is known from [MT2] [V] [TT] that $I^{(t)}$ (resp. I^t) is Cohen-Macaulay for some $t \geq 3$ (and then for all $t \geq 1$) if and only if I is the Stanley-Reisner ideal of a matroid complex (resp. a complete intersection Stanley-Reisner ideal) for a squarefree monomial ideal I.

There are some equivalent ways to define a graded ring is level, but we shall use the following definition. The ring S/I is called a level ring (for shortly, I level) if S/I is Cohen-Macaulay and the last free module in the minimal graded free resolution of S-module S/I has a basis of the same degree. The concept of a level ring was firstly introduced by R. Stanley. The level property is weaker than the Gorenstein property. A level ring of type 1 is precisely a Gorenstein ring. Level rings have attracted a lot of

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attention as in the work of M. Boij ([B]), T. Hibi ([H]), A. Geramita et. al. ([GHMS]), but many fundamental questions about this class of rings are still open.

In this article we shall give a complete answer of the above question for $t \geq 3$. Namely, we prove the following theorem:

Theorem 1. Let $I = I_\Delta$ be the Stanley-Reisner ideal of a simplicial complex Δ. Then, the following conditions are equivalent:

(1) I^t is level for all $t \geq 1$;
(2) I^t is level for some $t \geq 3$;
(3) $I^{(t)}$ is level for all $t \geq 1$;
(4) $I^{(t)}$ is level for some $t \geq 3$;
(5) I is a complete intersection and equi-generated.

For $t \geq 3$, the level properties of the ordinary power I^t and the symbolic one $I^{(t)}$ are equivalent, that is different from Cohen-Macaylay case.

For $t = 2$, the situation is quite complicated. Hence we consider the case that a simplicial complex Δ has dimension one. The ordinary power I_Δ^2 is level if and only if Δ is one of the following simplicial complexes: a 2-vertex segment, a 3-vertex segment, a triangle, a quadrilateral, and a pentagon. It follows from the fact that I_Δ^2 is level if and only if Δ is one of the above simplicial complexes in [MT1].

For the symbolic power case, we only give an answer when I is the Stanley-Reisner ideal of a one-dimensional matroid complex Δ. In this case, we think of the facets of Δ as the edges of a simple graph on the vertex set $[n]$. In other words, I is the Stanley-Reisner ideal of a matroid graph. Note that there are non-matroid graphs of which the second symbolic power of the Stanley-Reisner ideals are level. See the last two examples of the paper.

Theorem 2. Let I be the Stanley-Reisner ideal of a matroid graph Δ. Then, $I^{(2)}$ is level if and only if Δ is either a complete graph or a complete bipartite graph.

Now we explain the organization of the paper. In Section 2, we recall some notations and basic facts about the Stanley-Reisner ideal and matroids. Section 3 contains results for non-vanishing reduced homology groups which are used later. Section 4 is devoted to the proof of Theorem 1. After, Theorem 2 is proved in the last section.

2. Preliminaries

We will use some notation on graphs according to [D]. We refer the reader to e. g. [BH], [S],[MS] for the detailed information about combinatorial and algebraic background.

Let Δ be a simplicial complex on $[n] = \{1, \dots, n\}$ that is a collection of subsets of $[n]$ closed under taking subsets. We put $\dim F = |F| - 1$, where $|F|$ is the cardinality of F, and $\dim \Delta = \max\{\dim F \mid F \in \Delta\}$, which is called the dimension of Δ. It is clear that Δ can be uniquely determinate by the set of its maximal elements under inclusion, called by facets. The set of all facets of Δ is denote by $\mathfrak{F}(\Delta)$. The complex Δ is said pure if all its facets have the same cardinality.

For fixed field K, the i-th reduced simplicial (co)homology group of Δ denoted by $\widetilde{H}_i(\Delta; K)$ (w. r. t $\widetilde{H}^i(\Delta; K)$). Note that $\widetilde{H}_i(\Delta; K) = 0$ for all $i \in \mathbb{Z}$ if Δ is a cone (i.e., there exists a vertex x such that $x \in F$ for any facet F of Δ).

A matroid M on the ground set $[n]$ is a collection $\mathfrak{F}$ of subsets of $[n]$, which are called independent sets, satisfying the following conditions:

(i) $\emptyset \in \mathfrak{F}$,
(ii) If $I \in \mathfrak{F}$ and $J \subseteq I$, then $J \in \mathfrak{F}$,
(iii) If $I, J \in \mathfrak{F}$ and $|J| < |I|$, then there exists an element $x \in I \setminus J$ such that $J \cup \{x\} \in \mathfrak{F}$.

Maximal independent sets of M are called bases. They have the same cardinality called the rank of M. Denote by $\mathfrak{B}(M)$ the set of all bases of M. A dependent set is a subset of E which is not in $\mathfrak{F}$. Minimal dependent sets are called circuits of M. Denote by $\mathfrak{C}(M)$ the set of all circuits of M. It is clear that $\mathfrak{C}(M)$ determines M: $\mathfrak{F}$ consists of subsets of E that do not contain any member of $\mathfrak{C}(M)$.

It is apparent from the definition that the collection of independent sets of a matroid M forms a simplicial complex, which is called the matroid complex (or the independence complex) of M. This one is a pure simplicial complex of dimension $r(M) - 1$. For simlicity, we also use $\mathfrak{C}(\Delta)$, $\mathfrak{B}(\Delta)$ as the set of circuits and the set of bases of a matroid Δ.

We will also need the following property of a matroid due to by Stanley.

Lemma 2.1 (S, Theorem 3.4)**.** *Let Δ be a matroid complex. Then, Δ is a cone if and only if Δ is acyclic (i.e., has vanishing reduced homology).*

Suppose $V_1 \cap V_2 = \emptyset$. Let Δ_1 (respectively Δ_2) be a simplicial complex on V_1 (respectively V_2). Then, the simplicial join of Δ_1 and Δ_2, denoted by $\Delta_1 * \Delta_2$, is defined by $\{F \cup G \mid F \in \Delta_1, G \in \Delta_2\}$. It is clear that it is a simplicial complex on $V_1 \cup V_2$. The following lemma is easy to check from the definition.

Lemma 2.2. *If Δ_1, Δ_2 be two matroid complexes, which are not cones, over disjoint ground sets V_1, V_2 then so is $\Delta_1 * \Delta_2$ with the ground set $V_1 \cup V_2$.*

For a face $F \in \Delta$, we define the link and the star of F in a simplicial complex Δ to be

$$\operatorname{lk}_\Delta F = \{G \in \Delta \mid F \cup G \in \Delta, F \cap G = \emptyset\};$$

$$\operatorname{st}_\Delta F = \{G \in \Delta \mid F \cup G \in \Delta\}.$$

The next lemma appeared in [MTr, Lemma 2.3], and we would like to sketch the proof just for completeness.

Lemma 2.3. *Let Δ be a matroid complex which it is not a cone. If $\operatorname{lk}_\Delta(F) \neq \emptyset$ for some F, then it is also a matroid complex and is not a cone.*

Proof. It suffices to prove the case $F = \{x\}$ for $x \in V$. It is well-known that $\mathrm{lk}_\Delta(x)$ is a matroid. Assume the contrary, that $\mathrm{lk}_\Delta(x) \neq \emptyset$ is a cone for some $x \in V$. Let y be a center of this cone. Obviously, $y \neq x$. Since Δ is not a cone, there exists $B \in \mathfrak{F}(\Delta)$ such that $y \notin B$ (i.e. $x \notin B$). Put $F \in \mathfrak{F}(\mathrm{lk}_\Delta(x))$, then $F \cup \{x\} \in \mathfrak{F}(\Delta), x \notin F$. Therefore, $F' = (F \cup \{x\}) \setminus \{y\} \in \Delta$ and $|(F \cup \{x\}) \setminus \{y\}| < |B|$. By the definition of matroids, there exists $z \in B \setminus F'$ such that $F' \cup \{z\} \in \mathfrak{F}(\Delta)$. Thus, $(F' \cup \{z\}) \setminus \{x\} \in \mathfrak{F}(\mathrm{lk}_\Delta(x))$ and $y \notin (F' \cup \{z\}) \setminus \{x\}$, which is a contradiction. □

Let

$$\mathrm{core}([n]) = \{i \in [n] \mid \mathrm{st}_\Delta(i) \neq \Delta\},$$

and $\mathrm{core}(\Delta) = \Delta[\mathrm{core}([n])]$. It is clear that $\Delta[[n] \setminus \mathrm{core}([n])]$ is a simplex and $\{x_i \mid i \in [n] \setminus \mathrm{core}([n])\}$ forms a linear regular sequence of $S/I^{(t)}$. Therefore, $I^{(t)}$ is level if and only if $I^{(t)}_{\mathrm{core}(\Delta)})$ is level. For simplicity of exposition, throughout the rest of this paper, we always assume $\Delta = \mathrm{core}(\Delta)$, i.e. Δ is not a cone.

3. Non-vanishing reduced homology groups

Let Δ be a matroid complex of dimension $(d-1) \geq 0$. We shall give some non-vanishing reduced homology groups of certain subcomplexes of Δ, which are used later. The first result is as follows.

Theorem 3.1. *For any circuit $C \in \mathfrak{C}(\Delta)$,*

$$\widetilde{H}_{d-1}(\bigcup_{i \in C} \mathrm{st}_\Delta(C \setminus \{i\}); K) \neq 0.$$

Proof. Since $C \in \mathfrak{C}(\Delta)$, $C \setminus \{i\} \in \Delta$ for any $i \in C$, i.e. $\mathrm{st}_\Delta(C \setminus \{i\})) \neq \emptyset$. It is well known that the sub-complex $\Delta[C]$ is also matroid complex with its facet set $\{C \setminus \{i\} \mid i \in C\}$. This implies that $\Delta[C]$ is always not a cone. Fix $i \in C$, take $B \in \mathrm{lk}_\Delta(C \setminus \{i\})$. By the third condition of a matroid, $B \cup (C \setminus \{j\}) \in \Delta$ for all $j \in C$. Thus,

$$\bigcup_{i \in C} \mathrm{st}_\Delta(C \setminus \{i\}) = \Delta[C] * \mathrm{lk}_\Delta(C \setminus \{i\}).$$

Combining Lemma 2.3 and Lemma 2.2, $\bigcup_{i \in C} \mathrm{st}_\Delta(C \setminus \{i\})$ is always a matroid complex and is not a cone. Then, our assertion comes from Lemma 2.1. □

Next, we obtain the second result that:

Theorem 3.2. *Assume every circuit of Δ has the same cardinality and there exist two circuits of Δ which have at least one common vertex. Choose $C \neq C' \in \mathfrak{C}(\Delta)$ such that $|C \cap C'|$ is as large as possible. Then,*

$$\widetilde{H}_{d-1}\big(\bigcup_{U \subseteq (C \cup C'), |U|=2} \mathrm{st}_\Delta(C \cup C' \setminus U); K\big) \neq 0.$$

Proof. Let $W = C \cap C'$, $V_0 = C \setminus W$ and $V_0' = C' \setminus W$. Then, $|W| \geq 1$ and $|V_0| = |V_0'| = \alpha \geq 1$. Now, we need to prepare the following claims.

Claim 1: For any $x \in W$, there exists $W_x \subseteq W$ such that $|W_x| = \alpha$, $x \in W_x$ and

$$C_x = (V_0 \cup V_0' \cup W) \setminus W_x \in \mathfrak{C}(\Delta).$$

By a basic property of a matroid (see [O, Proposition 1.4.11]), there exists $C'' \in \mathfrak{C}(\Delta)$ such that $C'' \subseteq (C \cup C') \setminus \{x\}$. Let $U_1 = W \cap C''$, $U_2 = (C \cap C'') \setminus U_1$ and $U_3 = (C' \cap C'') \setminus U_1$. It yields that $x \in W \setminus U_1$. It is noticed that

$$\begin{aligned}
|C| &= |U_1| + |U_2| + |W \setminus U_1| + |C \setminus (C' \cup C'')| \\
|C'| &= |U_1| + |U_3| + |W \setminus U_1| + |C' \setminus (C \cup C'')| \\
|C''| &= |U_1| + |U_2| + |U_3|,
\end{aligned}$$

and $|C'' \cap C| = |U_1| + |U_2|$, $|C'' \cap C'| = |U_1| + |U_3|$. By choosing of C, C', $|U_2| \leq |W \setminus U_1|$ and $|U_3| \leq |W \setminus U_1|$. From this and our assumption, one can see that $C \setminus (C' \cup C'') = C' \setminus (C \cup C'') = \emptyset$ and $|U_2| = |U_3| = |W \setminus U_1|$. Put $W_x = W \setminus U_1$ and $C_x = C''$, we will obtain the result as required of this Claim.

Claim 2: For any $x, y \in W$, then either $W_x = W_y$ or $W_x \cap W_y = \emptyset$.

Assume the contrary, that $W_x \cap W_y \neq \emptyset$ and $W_x \neq W_y$ for some $x, y \in W$. As in the above Claim,

$$\begin{aligned}
C_x &= (V_0 \cup V_0' \cup W) \setminus W_x \in \mathfrak{C}(\Delta), \\
C_y &= (V_0 \cup V_0' \cup W) \setminus W_y \in \mathfrak{C}(\Delta).
\end{aligned}$$

Therefore, $C_x \neq C_y$ and

$$|C_x \cap C_y| = |V_0| + |V_0'| + |W| - |W_x| - |W_y| + |W_x \cap W_y| > |W|,$$

which is a contradiction with choosing C and C'.

By Claim 2, we have a partition of W by W_i for $i = 1, \ldots, s$. For simplicity, we rewrite $W_0 = V_0$ and $W_{s+1} = V_0'$. Then, $C \cup C'$ is a disjoint union of W_i for $i = 0, \ldots, s+1$. And, for all i, $|W_i| = \alpha$ and

$$(C \cup C') \setminus W_i \in \mathfrak{C}(\Delta).$$

Claim 3: For any $U = \{x, y\} \subseteq C \cup C'$, then $(C \cup C') \setminus U \in \Delta$ if and only if x, y belong to two different subsets W_i for some $i = 0, \ldots, s+1$.

It is clear that if $x, y \in W_i$ for some $i = 0, \ldots, s+1$ then $(C \cup C') \setminus U \notin \Delta$ by $(C \cup C') \setminus W_i \in \mathfrak{C}(\Delta)$. Assume $x \in W_a$, $y \in W_b$ for some $0 \leq a \neq b \leq s+1$ and $(C \cup C') \setminus U \notin \Delta$. Therefore, there exists a circuit C'' of M such that $C'' \subseteq (C \cup C') \setminus U$. Let $\alpha_i = |W_i \setminus C''| \geq 0$ for all i. It is noted that $\alpha_a \geq 1$ and $\alpha_b \geq 1$. Then,

$$\sum_{i=0}^{s+1} \alpha_i = \alpha,$$

by C'' has the same cardinality with C, i.e. $|C''| = (s+1)\alpha$. Thus, $((C \cup C') \setminus W_a) \neq C''$, and we have

$$\begin{aligned} |((C \cup C') \setminus W_a) \cap C''| &= \sum_{i \neq a} |W_i \cap C''| \\ &= \sum_{i \neq a} (\alpha - \alpha_i) \\ &= (s+1)\alpha - \sum_{i \neq a} \alpha_i = s\alpha + \alpha_a > s\alpha = |C \cap C'|, \end{aligned}$$

a contradiction.

We now return to prove our statement. Using Claim 3,

$$\bigcup_{U \subseteq (C \cup C'), |U|=2} \mathrm{st}_\Delta(C \cup C' \setminus U) = \bigcup_{x \in W_a, y \in W_b, a \neq b} \mathrm{st}_\Delta(C \cup C' \setminus \{x, y\}).$$

Also by this Claim, $\Delta[C \cup C']$ is a matroid complex with the facet set which consists of $C \cup C' \setminus \{x, y\}$ for x, y belong to two different subsets W_i. It implies that this complex is always neither emptyset nor a cone. Fix $x \in W_0$ and $y \in W_1$. Take any $B \in \mathrm{lk}_\Delta(C \cup C' \setminus \{x, y\})$. Then, by the third condition of a matroid, $B \in \mathrm{lk}_\Delta(C \cup C' \setminus \{x', y'\})$ for any x', y' belong to two different subsets W_i for some $i = 0, \ldots, s+1$. From this,

$$\bigcup_{U \subseteq (C \cup C'), |U|=2} \mathrm{st}_\Delta(C \cup C' \setminus U) = \Delta[C \cup C'] * \mathrm{lk}_\Delta(C \cup C' \setminus \{x, y\}).$$

Then, our statement comes from combining Lemmas 2.1, 2.2 and 2.3. □

4. Large symbolic powers

First, we need to recall a formula for computing the multigraded Betti numbers of a monomial ideal due to by Miller and Sturmfels throughout the (non)-vanishing of reduced homology groups of certain simplicial complexes. Let $\mathbf{e}_i$ be the i^{th}-unit vector for $i = 1, \ldots, n$. For each vector $\mathbf{a} \in \mathbb{N}^n$, define $\mathbf{e}_{\mathrm{supp}(\mathbf{a})} = \sum_{i \in \mathrm{supp}(\mathbf{a})} \mathbf{e}_i$, where $\mathrm{supp}(\mathbf{a}) = \{i \mid a_i \neq 0\}$. Given a monomial ideal J and a degree $\mathbf{a} \in \mathbb{N}^n$, the lower Koszul simplicial complex of S/J in degree $\mathbf{a}$ is

$$K_\mathbf{a}(J) = \{F \subseteq \mathrm{supp}(\mathbf{a}) \mid \mathbf{x}^{\mathbf{a} - \mathbf{e}_{\mathrm{supp}(\mathbf{a})}}.\mathbf{x}^F \notin J\},$$

where $\mathbf{x}^F = \prod_{i \in F} x_i$ and $\mathbf{x}^\mathbf{a} = \prod_{i \in \mathrm{supp}(\mathbf{a})} x_i^{a_i}$.

Theorem 4.1 (MS, Theorem 5.11)**.** *Given a vector $\mathbf{a} \in \mathbb{N}^n$ with support $\mathrm{supp}(\mathbf{a})$ and a monomial ideal J in S, the Betti numbers of S/J in degree $\mathbf{a}$ can be expressed as*

$$\beta_{i,\mathbf{a}}(S/J) = \dim_K(\widetilde{H}^{|\,\mathrm{supp}(\mathbf{a})|-i-1}(K_\mathbf{a}(J); K)) = \dim_K(\widetilde{H}_{|\,\mathrm{supp}(\mathbf{a})|-i-1}(K_\mathbf{a}(J); K)),$$

for all i.

From the level property of a symbolic power for $t \geq 2$, we always obtain the condition that the original ideal is equi-generated as follows.

Theorem 4.2. *Let Δ be the matroid complex of dimension $(d-1) \geq 0$ and I be the Stanley-Reisner ideal of Δ. If $S/I^{(t)}$ is level for some $t \geq 2$, then I is equi-generated, i.e. every circuit of Δ has the same cardinality.*

Proof. For each circuit $C \in \mathfrak{C}(\Delta)$, let $\mathbf{a}_C = \sum_{i \in C} t\mathbf{e}_i + \sum_{i \notin C} \mathbf{e}_i$. Then,

$$K_{\mathbf{a}_C}(I^{(t)}) = \{F \subseteq [n] \mid f_C.\mathbf{x}^F \notin I^{(t)}\},$$

where $f_C = \prod_{i \in C} x_i^{t-1}$. For each $B \in \mathfrak{B}(\Delta)$, one can see that $|C \setminus B| \geq 1$. This implies that $f_C.\mathbf{x}^F \notin I^{(t)}$ if and only if $F \subseteq B$ for some $B \in \mathfrak{B}(\Delta)$ such that $|C \setminus B| = 1$. Therefore,

$$K_{\mathbf{a}_C}(I^{(t)}) = \bigcup_{i \in C} \mathrm{st}_\Delta(C \setminus \{i\}).$$

Using Theorem 4.1 and Theorem 3.1,

$$\beta_{n-d,\mathbf{a}_C}(S/I^{(t)}) = \dim_K(\widetilde{H}_{d-1}(\bigcup_{i \in C} \mathrm{st}_\Delta(C \setminus \{i\}); K)) \neq 0.$$

This yields $\beta_{n-d,(t-1)|C|+n}(S/I^{(t)}) \neq 0$ for each $C \in \mathfrak{C}(\Delta)$. By our assumption, every circuit of Δ has the same cardinality as required. $\square$

We are now in a position to prove the first main result of this paper.

Theorem 4.3. *Let Δ be a simplicial complex of dimension $d-1 \geq 0$ and I be the Stanley-Reisner ideal of Δ. Then, the following conditions are equivalent:*

(1) *S/I^t is level for all $t \geq 1$,*
(2) *S/I^t is level for some $t \geq 3$,*
(3) *$S/I^{(t)}$ is level for all $t \geq 1$,*
(4) *$S/I^{(t)}$ is level for some $t \geq 3$,*
(5) *I is equi-generated and a complete intersection.*

Proof. The implications (1) $\Rightarrow$ (2) and (3) $\Rightarrow$ (4) are clear. Note that for some $t \geq 1$ S/I^t is Cohen-Macaulay if and only if $S/I^{(t)}$ is Cohen-Macaulay and $I^t = I^{(t)}$. Hence S/I^t is level if and only if $S/I^{(t)}$ is level and $I^t = I^{(t)}$. Then the implications (1) $\Rightarrow$ (3) and (2) $\Rightarrow$ (4) are clear.

We consider the implication (5) $\Rightarrow$ (1). The t-th power of the graded maximal ideal has a t-linear resolution. See, e.g., [BH, Exercises 4.1.17]. Hence if I is equi-generated and a complete intersection, then I^t has a pure resolution, since each pair of generators of I is coprime and has the same degree. Since S/I^t is Cohen-Macaulay, it is level.

Now it is enough to prove that (4) implies (5). By Theorem 4.2, we only need to show that two different circuits of Δ must be disjoint. Assume the contrary, that there exist two circuits of Δ which have at least a common vertex. Choose

$C \neq C' \in \mathfrak{C}(\Delta)$ such that cardinality of $\emptyset \neq W = C \cap C'$ is as large as possible. Let $\mathbf{a}_{(C,C')} = \sum_{i \in C}(t-1)\mathbf{e}_i + 2\sum_{i \in C' \setminus C} \mathbf{e}_i + \sum_{i \notin C \cup C'} \mathbf{e}_i$. Then,

$$K_{\mathbf{a}_{(C,C')}}(I^{(t)}) = \{F \subseteq [n] \mid f_{(C,C')}.\mathbf{x}^F \notin I^{(t)}\},$$

where $f_{(C,C')} = \prod_{i \in C} x_i^{t-2} \prod_{i \in C' \setminus C} x_i$. For each $B \in \mathfrak{B}(\Delta)$, one can see that $|C \setminus B| \geq 1$ and $|C' \setminus B| \geq 1$.

If $|(C \cup C') \setminus B| = 1$, assume $x \in (C \cup C') \setminus B$, then x must belong to W and $(C \cup C') \setminus \{x\} \subseteq B$. Since Claim 1 in the Theorem 3.2, there exists $x \in W_x \subseteq W$ such that $C_x = (V_0 \cup V_0' \cup W) \setminus W_x \in \mathfrak{C}(\Delta)$, which is a contradiction by $C_x \subsetneq B \in \Delta$.

If $|(C \cup C') \setminus B| \geq 3$, then $f_{(C,C')} \in P_B^t$ by $t \geq 3$. Therefore, $f_{(C,C')}.\mathbf{x}^F \notin I^{(t)}$ if and only if $F \subseteq B$ for some $B \in \mathfrak{B}(\Delta)$ such that either $|(C \cup C') \setminus B| = 2$ if $t = 3$ or $(C \cup C') \setminus B = \{x, y\}$ for $x \in C, y \in C' \setminus C$ if $t \geq 4$.

We consider two cases as follows.

Case 1: $t = 3$. Then, as in the above,

$$K_{\mathbf{a}_{(C,C')}}(I^{(t)}) = \bigcup_{U \subseteq (C \cup C'), |U|=2} \mathrm{st}_\Delta(C \cup C' \setminus U).$$

Using Theorem 3.2, $\widetilde{H}_{d-1}(K_{\mathbf{a}_{(C,C')}}(I^{(t)}); K) \neq 0$.

Case 2: $t \geq 4$. We can see that

$$K_{\mathbf{a}_{(C,C')}}(I^{(t)}) = \bigcup_{x \in C, y \in (C' \setminus C)} \mathrm{st}_\Delta(C \cup C' \setminus \{x, y\}).$$

Similarly as in the proof of Theorem 3.2, fixed $x \in C, y \in C' \setminus C$, one can check that

$$\bigcup_{x \in C, y \in (C' \setminus C)} \mathrm{st}_\Delta(C \cup C' \setminus \{x, y\}) = \Delta[C] * \Gamma * \mathrm{lk}_\Delta(C \cup C' \setminus \{x, y\})$$

where Γ is the matroid complex which consists of all subsets $(C' \setminus C) \setminus \{z\}$ for $z \in C' \setminus C$. Using again Lemma 2.1, Lemma 2.3 and Lemma 2.2, $\widetilde{H}_{d-1}(K_{\mathbf{a}_{(C,C')}}(I^{(t)}); K) \neq 0$.

From both of cases and Theorem 4.1, one can see that $\beta_{n-d,(t-1)|C|+n-|W|}(S/I^{(t)}) \neq 0$. Combining it and Theorem 4.2, we will obtain a contradiction with the levelness of $S/I^{(t)}$. ☐

It can be noted that there is a Stanley-Reisner ideal I such that $S/I^{(2)}$ is level but S/I^2 is not (see the last example of next section). So, $t = 3$ is the best value for this theorem.

Corollary 4.4. *Let Δ be a simplicial complex and I be the Stanley-Reisner ideal of Δ. Then, the following conditions are equivalent:*

(1) *S/I^t is Gorenstein for all $t \geq 1$,*
(2) *S/I^t is Gorenstein for some $t \geq 3$,*
(3) *$S/I^{(t)}$ is Gorenstein for all $t \geq 1$,*
(4) *$S/I^{(t)}$ is Gorenstein for some $t \geq 3$,*

(5) *I is a principal ideal.*

Proof. The implications (1) $\Rightarrow$ (2), (2) $\Rightarrow$ (4), (1) $\Rightarrow$ (3), (3) $\Rightarrow$ (4) and (5) $\Rightarrow$ (1) are clear. Hence it is enough to prove that (4) implies (5). Assume the condition (4). By Theorem 4.3, I is equi-generated and a complete intersection. Suppose I is not principal. Suppose I is minimally generated by p monomials for $p \geq 2$. Set $J = (x_1, x_2, \ldots, x_p)$. Then for $t \geq 3$, J^t is not Gorenstein, since the coefficient of the highest degree of the numerator of Hilbert series of S/J^t is $\binom{p+t-2}{t-1} \neq 1$. Hence I^t is not Gorenstein, which is a constradiction with the condition (4). $\square$

5. The second symbolic power

In this section we only consider the second symbolic power of Stanley-Reisner ideal of a one-dimensional matroid complex. For simplicity of exposition, in this section, we assume that Δ is a matroid complex of dimension one. Then, $S/I_\Delta^{(2)}$ is Cohen-Macaulay of dimension two. It is clear that Δ can be viewed as a simple graph on $[n]$ for $n \geq 2$. It can be noted that if $n = 2, 3$ then Δ is a complete graph and I_Δ is a principal ideal, so $I_\Delta^{(2)}$ is always level. So, we may assume that $n \geq 4$.

For the proof of the main theorem, some more preparations are needed.

Lemma 5.1. *If Δ does not contain any triangles then Δ is a complete bipartite graph.*

Proof. By the connectedness of Δ, one may assume that $12, 13 \in \Delta$. Let

$$X = \{i \in [n] \mid i \neq 2, 2i \in \Delta\}$$

and

$$Y = \{j \in [n] \mid j \neq 1, \text{ there exists a vertex } i_j \in X \text{ such that } ji_j \in \Delta\}.$$

It is clear that $1 \in X$ and both of $2, 3$ are in Y. Firstly, for all $a \neq b \in X$, then $ab \notin \Delta$ by the triangle-free property of Δ. Take $a \neq b \in Y$, then there exist $i_a, i_b \in X$ such that $ai_a, bi_b \in \Delta$. If $i_a = i_b$ then $ab \notin \Delta$ as above. If $i_a \neq i_b$ then $i_a i_b \notin \Delta$. Therefore, $i_a b \in \Delta$ by the matroid condition. Thus, $ab \notin \Delta$. Secondly, take any vertex $u \in [n] \setminus \{1, 2, 3\}$, one may see that either $1u$ or $2u$ is in Δ by the matroid property. Therefore, $X \cup Y = [n]$ and it can check that $X \cap Y = \emptyset$. Take any $u \in X, v \in Y$. If $v = 2$ then $uv \in \Delta$. If $v \neq 2$ then there exists $i(v) \in X$ such that $vi_v \in \Delta$. If $i_v = u$ then $uv \in \Delta$, otherwise $i_v \neq u$ then $uv \in \Delta$ by its matroid property. Thus, uv always belongs to Δ which implies that Δ is the complete bipartite graph over X and Y as required.

$\square$

Proposition 5.2. *If Δ be a complete graph then $I_\Delta^{(2)}$ is level.*

Proof. Let $\mathbf{a} = 2(\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3) + \sum_{i=4}^{n} \mathbf{e}_i$. Then, $\mathrm{supp}(\mathbf{a}) = [n]$ and by definition,

$$K_{\mathbf{a}}(I_\Delta^{(t)}) = \{F \subseteq [n] \mid x_1 x_2 x_3 . \mathbf{x}^F \notin I_\Delta^{(2)} . \}$$

Note that, if $|F \setminus \{1,2,3\}| \geq 1$ then $x_1x_2x_3.\mathbf{x}^F \in I_\Delta^{(2)}$. If $F \subseteq \{1,2,3\}$ then one can see that the facets of $K_{\mathbf{a}}(I_\Delta^{(t)})$ are $12, 23, 31$. Therefore, by Theorem 4.1, $\beta_{n-2,\mathbf{a}}(S/I_\Delta^{(t)}) = \dim(\widetilde{H}_1(K_{\mathbf{a}}(I_\Delta^{(t)}); K)) = \dim(\widetilde{H}_1(\mathbb{S}^1; K)) \neq 0$. It is enough to show that $\widetilde{H}_{|\operatorname{supp}(\mathbf{b})|-n+1}(K_{\mathbf{b}}(I_\Delta^{(2)}); K) = 0$ for all $\mathbf{b} \in \mathbb{N}^n$ and $|\mathbf{b}| \neq n+3$. Fix a vector $\mathbf{b} \in \mathbb{N}^n$ with $|\mathbf{b}| \neq n+3$, let $W = \operatorname{supp}(\mathbf{b}), \mathbf{u} = \mathbf{b} - \mathbf{e}_{\operatorname{supp}(\mathbf{b})}$. Let

$$\Delta_{\mathbf{u}} = \{F \subseteq [n] \mid \mathbf{x}^{\mathbf{u}}.\mathbf{x}^F \notin I_\Delta^{(2)}\},$$

then $K_{\mathbf{b}}(I_\Delta^{(2)}) = \Delta_{\mathbf{u}}[W]$. It is clear that $\operatorname{supp}(\mathbf{u}) \subseteq W$. We distinguish some types of $\Delta_{\mathbf{u}}$.

Type 1: $|\operatorname{supp}(\mathbf{u})| \geq 4$. It is clear that $\mathbf{x}^{\mathbf{u}} \in I_\Delta^{(2)}$. Therefore, $\Delta_{\mathbf{u}} = \emptyset$.

Type 2: $|\operatorname{supp}(\mathbf{u})| = 3$. Write $1, 2, 3 \in \operatorname{supp}(\mathbf{u})$.

(i) If $u_1 = u_2 = u_3 = 1$ then the facets of $\Delta_{\mathbf{u}}$ are $12, 13, 23$;
(ii) If $u_1 \geq 2, u_2 = u_3 = 1$ then the facets of $\Delta_{\mathbf{u}}$ are $12, 13$;
(iii) If $u_1 \geq 2, u_2 \geq 2, u_3 = 1$ then the facets of $\Delta_{\mathbf{u}}$ are 12;
(iv) If $u_1 \geq 2, u_2 \geq 2, u_3 \geq 2$ then $\Delta_{\mathbf{u}} = \emptyset$ by $x_1^2x_2^2x_3^2 \in I_\Delta^{(2)}$.

Type 3: $|\operatorname{supp}(\mathbf{u})| = 2$. Write $1, 2 \in \operatorname{supp}(\mathbf{u})$. If $|F \setminus \{1,2\}| \geq 2$ then $\mathbf{x}^{\mathbf{u}}.\mathbf{x}^F \in I_\Delta^{(2)}$. Note that $\mathbf{x}^{\mathbf{u}}.x_i \notin P_{1,2}^2$ for all i. Therefore, the facets of $\Delta_{\mathbf{u}}$ are $\{12i \mid i = 3, \ldots, n\}$.

Type 4: $|\operatorname{supp}(\mathbf{u})| = 1$. Write $1 \in \operatorname{supp}(\mathbf{u})$. If $|F \setminus \{1\}| \geq 3$ then $\mathbf{x}^{\mathbf{u}}.\mathbf{x}^F \in I_\Delta^{(2)}$. From $\mathbf{x}^{\mathbf{u}}.x_ix_j \notin P_{1,i}^2$ for all $i \neq j$, the facets of $\Delta_{\mathbf{u}}$ are $\{1ij \mid 2 \leq i < j \leq n\}$.

Type 5: $|\operatorname{supp}(\mathbf{u})| = 0$. One can see that the facets of $\Delta_{\mathbf{u}}$ are $\{ijh \mid 1 \leq i < j < h \leq n\}$.

From these types and $\operatorname{supp}(\mathbf{u}) \subseteq W$, we always obtain $\widetilde{H}_{|W|-n+1}(\Delta_{\mathbf{u}}[W]; K)) = 0$ except the case type 2 (i) occurs and $|W| = n$, i.e. $|\mathbf{b}| = n+3$. From this, we obtain as required. □

Proposition 5.3. *If Δ is a complete bipartite graph then $I_\Delta^{(2)}$ is level.*

Proof. Assume that Δ is a complete bipartite graph $K_{|X|,|Y|}$ for $X \cup Y = [n], X \cap Y = \emptyset, X, Y \neq \emptyset$. Fix a vector $\mathbf{b} \in \mathbb{N}^n$, let $W = \operatorname{supp}(\mathbf{b}), \mathbf{u} = \mathbf{b} - \mathbf{e}_{\operatorname{supp}(\mathbf{b})}$. Let

$$\Delta_{\mathbf{u}} = \{F \subseteq [n] \mid \mathbf{x}^{\mathbf{u}}.\mathbf{x}^F \notin I_\Delta^{(2)}\},$$

then $K_{\mathbf{b}}(I_\Delta^{(2)}) = \Delta_{\mathbf{u}}[W]$. Similarly as in the above proof, we have some types of $\Delta_{\mathbf{u}}$.

Type 1: $|\operatorname{supp}(\mathbf{u})| \geq 4$. It is clear that $\mathbf{x}^{\mathbf{u}} \in I_\Delta^{(2)}$. Therefore, $\Delta_{\mathbf{u}} = \emptyset$.

Type 2: $|\operatorname{supp}(\mathbf{u})| = 3$. Write $1, 2, 3 \in \operatorname{supp}(\mathbf{u})$.

(i) If $1, 2, 3 \in X$ or $1, 2, 3 \in Y$ then $\Delta_{\mathbf{u}} = \emptyset$ by $x_1x_2x_3 \in I_\Delta^{(2)}$;
(ii) If $1, 2 \in X$ and $3 \in Y$ then the facets of $\Delta_{\mathbf{u}}$ are $23, 13$ if $u_1 = u_2 = u_3 = 1$, or 13 if $u_1 \geq 2, u_2 = u_3 = 1$, or 23 if $u_1 = 1, u_2 \geq 2, u_3 = 1$, or $\emptyset$ otherwise.

Type 3: $|\operatorname{supp}(\mathbf{u})| = 2$. Write $1, 2 \in \operatorname{supp}(\mathbf{u})$.

(i) If $1, 2 \in X$ or $1, 2 \in Y$ then $\Delta_{\mathbf{u}}$ is $\operatorname{st}_\Delta(1) \cup \operatorname{st}_\Delta(2)$ if $u_1 = u_2 = 1$, or $\operatorname{st}_\Delta(1)$ if $u_1 \geq 2, u_2 = 1$, or $\operatorname{st}_\Delta(2)$ if $u_1 = 1, u_2 \geq 2$, or $\emptyset$ otherwise.

(ii) If $1 \in X$ and $2 \in Y$ then the facets of $\Delta_{\mathbf{u}}$ are $\{12i \mid i = 3, \dots, n\}$ if $u_1 = u_2 = 1$, or $\{1i \mid i = 3, \dots, n\}$ if $u_1 \geq 2, u_2 = 1$, or $\{2i \mid i = 3, \dots, n\}$ if $u_1 = 1, u_2 \geq 2$, or $\emptyset$ otherwise.

Type 4: $|\operatorname{supp}(\mathbf{u})| = 1$. Write $1 \in \operatorname{supp}(\mathbf{u})$. Assume $1 \in X$, then the facets of $\Delta_{\mathbf{u}}$ are $\{1ij \mid i \in Y \text{ or } j \in Y\}$.

Type 5: $|\operatorname{supp}(\mathbf{u})| = 0$. One can see that the facets of $\Delta_{\mathbf{u}}$ are

$$\{ijh \mid \text{ except in the case of } i, j, h \in X \text{ or in the case of } i, j, h \in Y\}.$$

One can see that $\widetilde{H}_{|W|-n+1}(\Delta_{\mathbf{u}}[W]; K)) = 0$ if form of $\Delta_{\mathbf{u}}$ likes as type 1, type 2, type 3 (ii) and type 4 by $\operatorname{supp}(\mathbf{u}) \subseteq W$ and the acyclic property of a cone. We distinguish some cases as follows.

Case 1: $|X| = 1$ or $|Y| = 1$. Assume $|X| = 1$ and $t \in X$. Therefore, if $\Delta_{\mathbf{u}}$ has form as type 3 (i) $\widetilde{H}_{|W|-n+1}(\Delta_{\mathbf{u}}[W]; K)) \neq 0$ when $W = [n] \setminus \{t\}$ and $u_1 = u_2 = 1$ for $1, 2 \in Y$. In this case, $\Delta_{\mathbf{u}}[W]$ consists of two points $1, 2$. One can see that $\widetilde{H}_{|W|-n+1}(\Delta_{\mathbf{u}}[W]; K)) = 0$ if $\Delta_{\mathbf{u}}$ has form as type 5 because it is a cone over t.

Case 2: $|X| = 2$ and $|Y| = 2$. Then, I_Δ is a complete intersection which implies the level property of $I_\Delta^{(2)}$.

Case 3: $|X| \geq 2$ and $|Y| \geq 3$ or $|X| \geq 3$ and $|Y| \geq 2$. Assume $|X| \geq 2$ and $|Y| \geq 3$. If $\Delta_{\mathbf{u}}$ has form as type 3 (i) then $\widetilde{H}^{|W|-n+1}(\Delta_{\mathbf{u}}[W]; K)) \neq 0$ when $\mathbf{b}$ has a form $2(\mathbf{e}_1 + \mathbf{e}_2) + \sum_{i \geq 3} \mathbf{e}_n$ (for $1, 2 \in X$ or $1, 2 \in Y$). In this case $W = [n], \mathbf{u} = \mathbf{e}_1 + \mathbf{e}_2$ and the reduced cohomology groups are not vanishing by there exists a "empty" circle in $\Delta_{\mathbf{u}}[W]$.

In fact, if $|W| = n-2$ then $\Delta_{\mathbf{u}}[W] \neq \{\emptyset\}$ by it contains some points; if $|W| = n-1$ then $\Delta_{\mathbf{u}}[W]$ is always connected; if $|W| = n$ and either $u_1 \geq 2$ or $u_2 \geq 2$ then $\widetilde{H}_1(\Delta_{\mathbf{u}}[W]; K)) = 0$. If $\Delta_{\mathbf{u}}$ has form as type 5, then $\Delta_{\mathbf{u}}[W] \neq \{\emptyset\}$ if $|W| = n-2$ and $\Delta_{\mathbf{u}}[W]$ is connected if $|W| = n-1$. When $|W| = n$, by induction on $|X| \geq 1$ and the Mayer-Vietoris sequence, one can check that $\widetilde{H}_1(\Delta_{\mathbf{u}}; K) = 0$.

From these cases, $\beta_{n-2}((S/I_\Delta^{(2)}))$ only concentrated at degree $n+2$, which implies the conclusion as required. □

Proposition 5.4. *If Δ is neither a complete graph nor a complete bipartite graph then $I_\Delta^{(2)}$ is not level.*

Proof. By Lemma 5.1, Δ must contain at least a triangle, say $12, 23, 31 \in \Delta$. Put $\mathbf{a} = 2(\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3) + \sum_{i=4}^n \mathbf{e}_i$. Arguing as in the proof of Proposition 5.2, $\beta_{n-2,\mathbf{a}}(S/I_\Delta^{(2)}) \neq 0$. Because Δ is not a complete graph, we assume $14 \notin \Delta$. From the matroid property of Δ, $24, 34 \in \Delta$. Let $\mathbf{b} = 2(\mathbf{e}_1 + \mathbf{e}_4) + \mathbf{e}_2 + \mathbf{e}_3 + \sum_{i>4}^n \mathbf{e}_i$ then $\operatorname{supp}(\mathbf{b}) = [n]$ and $|b| = n+2$. Then,

$$K_{\mathbf{b}}(I_\Delta^{(2)}) = \{F \subseteq [n] \mid x_1 x_4.\mathbf{x}^F \notin I_\Delta^{(2)}\} = \operatorname{st}_\Delta(1) \cup \operatorname{st}_\Delta(4).$$

We can rewrite $\operatorname{st}_\Delta(1) \cup \operatorname{st}_\Delta(4) = \Delta_1 \cup \Delta_2$, where the facets of Δ_1 are $12, 13, 24, 34$ and the facets of Δ_2 are the other facets of $\operatorname{st}_\Delta(1) \cup \operatorname{st}_\Delta(4)$. Therefore, $\dim(\Delta_1 \cap \Delta_2) \leq 0$.

Then, $\widetilde{H}_1(\Delta_1 \cap \Delta_2; K) = 0$. And, it is clear that $\widetilde{H}_1(\Delta_1; K) \neq 0$. By using the Mayer-Vietoris sequence, $\cdots \to \widetilde{H}_1(\Delta_1 \cap \Delta_2; K) \to \widetilde{H}_1(\Delta_1; K) \oplus \widetilde{H}_1(\Delta_2; K) \to \widetilde{H}_1(\Delta_1 \cup \Delta_2; K) \to \widetilde{H}_0(\Delta_1 \cap \Delta_2; K) \to \cdots$, we have $\widetilde{H}_1(\Delta_1 \cup \Delta_2; K) \neq 0$. Thus, by Theorem 4.1,

$$\beta_{n-2,\mathbf{b}}(S/I_\Delta^{(2)}) = \dim_K(\widetilde{H}_1(K_{\mathbf{b}}(I_\Delta^{(2)}); K)) \neq 0.$$

This proves our assertion. □

Combining Proposition 5.2, Proposition 5.3 and Proposition5.4 yields the result as follows.

Theorem 5.5. *Let Δ be a matroid graph over $[n]$ for $n \geq 2$. Then, $I_\Delta^{(2)}$ is level if and only if Δ is either a complete graph or a complete bipartite graph.*

In the end of this section, we shall give two examples of non-matroid graphs of which the second symbolic power of the Stanley-Reisner ideals are level. These examples are inspired by computations of the computer algebra system as CoCoA [Co]. For the second example, it can be noted that the second ordinary power of its Stanley-Reisner ideal is not Cohen-Macaulay by [MT1, Corollary 3.4], so it is not also level.

Example 5.6. (1) *Let $n = 5$ and Δ be a pentagon such that its facet set is $\{12, 23, 34, 45, 15\}$. Then, $I_\Delta^{(2)}$ is level. This induced from the minimal graded resolution of $S/I_\Delta^{(2)}$ as follows:*

$$0 \to S(-6)^{10} \longrightarrow S(-5)^{24} \longrightarrow S(-4)^{15} \longrightarrow S \to 0.$$

(2) *Let $n = 10$ and Δ be the Petersen graph such that its facet set is*

$$\{12, 23, 34, 45, 15, 16, 27, 38, 49, 510, 68, 69, 79, 710, 810\}.$$

Then, $I_\Delta^{(2)}$ is level but I_Δ^2 is not level. In fact that, $S/I_\Delta^{(2)}$ has a minimal graded resolution that

$$0 \to S(-11)^{90} \longrightarrow S(-10)^{684} \longrightarrow S(-9)^{2240} \longrightarrow S(-8)^{4095} \longrightarrow S(-6)^5 \oplus S(-7)^{4500}$$
$$\longrightarrow S(-5)^{60} \oplus S(-6)^{2945} \longrightarrow S(-4)^{75} \oplus S(-5)^{1068} \longrightarrow S(-3)^{30} \oplus S(-4)^{165} \longrightarrow S \to 0.$$

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