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A strong convergence theorem for Tseng's extragradient method for solving variational inequality problems

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Abstract In this paper, we introduce a new algorithm for solving variational inequality problems with monotone and Lipschitz-continuous mappings in real Hilbert spaces. Our algorithm requires only to compute one projection onto the feasible set per iteration. We prove under certain mild assumptions, a strong convergence theorem for the proposed algorithm to a solution of a variational inequality problem. Finally, we give some numerical experiments illustrating the performance of the proposed algorithm for variational inequality problems.

Keywords Tseng's extragradient · viscosity method · variational inequality problem.

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1 Introduction

Let H be a real Hilbert space with the inner product $\langle\cdot,\cdot\rangle$ and the induced norm $\|\cdot\|$. Let C be a nonempty closed convex subset in H.

Now, we consider the classical *variational inequality problem* (**VIP**), which is to find a point $x^* \in C$ such that

$$\langle Ax^*, y - x^*\rangle \geq 0, \quad \forall y \in C, \tag{1}$$

where $A : H \to H$ is a mapping.

We assume that the following conditions hold:

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(C1) The solution set of (1), denoted by $VI(C,A)$, is nonempty.
(C2) The mapping A is monotone, i.e.,

$$\langle Ax - Ay, x - y \rangle \geq 0, \ \ \forall x, y \in H.$$

(C3) The mapping A is Lipschitz-continuous with constant $L > 0$, i.e., there exists $L > 0$ such that

$$\|Ax - Ay\| \leq L\|x - y\|, \ \forall x, y \in H.$$

Variational inequality theory is an important tool in economics, engineering mechanics, mathematical programming, transportation, and others (see, for example, [2,3,11,16,18]). Many numerical methods have been constructed for solving variational inequalities and related optimization problems, see [7,4,5,6,17,23,31,32,33,34,36,40,41] and the references therein.

One of the most popular method for solving the problem **(VIP)** is the extragradient method **(EGM)** which introduced in 1976 by Korpelevich [19] as follows:

$$\begin{cases} x \in C, \\ y_n = P_C(x_n - \lambda A x_n), \\ x_{n+1} = P_C(x_n - \lambda A y_n), \ \ \forall n \geq 0, \end{cases} \tag{2}$$

where $\lambda \in (0, \frac{1}{L})$ and P_C denotes the metric projection from H onto C.

First, this method **(EGM)** was proposed for solving saddle point problems in finite-dimensional space, but, after that, this method was further extended to the problem **(VIP)** in both Euclidean spaces and Hilbert spaces.

In recent years, the method **(EGM)** has received great attention by many authors, that is, there are many results have been obtained by the extragradient method and its modifications when A is monotone and L-Lipschitz continuous in infinite-dimensional Hilbert spaces (see, for instance, [8,22,26,39]).

In [22], to obtian the strong convergence of extragradient method **(EGM)** in real Hilbert spaces, Maingé proposed the algorithm as follows

$$\begin{cases} x_0 \in H, \\ y_n = P_C(x_n - \lambda_n A x_n), \\ t_n = P_C(x_n - \lambda_n A y_n), \\ x_{n+1} = t_n - \alpha_n F t_n, \end{cases}$$

where $A : H \to H$ is monotone on C and L-Lipschitz continuous on H and $F : H \to H$ is Lipschitz continuous and strongly monotone on C such that $VI(C,A) \neq \emptyset$. Maingé proved that if the parameters satisfy the conditions: $\lambda_n \in [a,b] \subset (0, \frac{1}{L})$, $\alpha_n \in [0,1)$, $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$ then the sequence $\{x_n\}$ converges strongly to $x^* \in VI(C,A)$, where x^* is the element such that

$$\langle Fx^*, x - x^* \rangle \geq 0 \ \forall x \in VI(C,A).$$

The drawback of these results is the need to calculate two projections onto the closed convex set C in each iteration. So, in case that the set C is not simple to calculate projection onto it, a minimum distance problem has to be solved twice in one iteration, which is a fact that might affect the efficiency and applicability of this method **(EGM)**.

To overcome this drawback. Censor et al. [4] introduced the subgradient extragradient method **(SEGM)**, in which the second projection onto C is replaced by a projection onto a specific constructible half-space:

$$\begin{cases} y_n = P_C(x_n - \lambda A x_n), \\ T_n = \{w \in H : \langle x_n - \lambda A x_n - y_n, w - y_n \rangle \leq 0\}, \\ x_{n+1} = P_{T_n}(x_n - \lambda A y_n), \quad \forall n \geq 0, \end{cases} \tag{3}$$

where $\lambda \in (0, \frac{1}{L})$. In order to obtain the strong convergence of the method **(SEGM)**, they [5] also introduced the following hybrid subgradient extragradient method **(HSEGM)**:

$$\begin{cases} x_0 \in H \\ y_n = P_C(x_n - \mu A x_n), \\ T_n = \{x \in H : \langle x_n - \lambda A x_n - y_n, x - y_n \rangle \leq 0\}, \\ z_n = \alpha_n x_n + (1 - \alpha_n) P_{T_n}(x_n - \lambda A y_n), \\ C_n = \{w \in H : \|z_n - w\| \leq \|x_n - w\|\}, \\ Q_n = \{w \in H : \langle x_n - w, x_0 - x_n \rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0, \quad \forall n \geq 0, \end{cases} \tag{4}$$

and they proved, under appropriate conditions, that the the sequence $\{x_n\}$ generated by (4) converges strongly to a point $p = P_{VI(C,A)}(x_0)$.

As Kraikaew and Saejung noted in [17] that the sequence $\{x_n\}$ generated by (4) seems to be difficult to use in practical problems because the computation of the next iterate becomes a subproblem of finding a point in the intersection of two additional half-spaces.

Inspired by the results in [5], Kraikaew and Saejung [17] combined the subgradient extragradient method and the Halpern method to propose an algorithm, which is called the Halpern subgradient extragradient method **(HPSEGM)** for solving the problem **(VIP)** as follows:

$$\begin{cases} x_0 \in H, \\ y_n = P_C(x_n - \lambda A x_n), \\ T_n = \{x \in H : \langle x_n - \lambda A x_n - y_n, x - y_n \rangle \leq 0\}, \\ z_n = P_{T_n}(x_n - \lambda A y_n), \\ x_{n+1} = \alpha_n x_0 + (1 - \alpha_n) z_n, \quad \forall n \geq 0, \end{cases} \tag{5}$$

where $\lambda \in (0, \frac{1}{L})$, $\{\alpha_n\} \subset (0,1)$, $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = +\infty$, and they proved that the sequence $\{x_n\}$ generated by the method **(HPSEGM)** converges strongly to a point $p = P_{VIP(C,A)} x_0$.

In this paper, we wish to focus on a close, but different type of the algorithm, known as *Tseng's extragradient method* **(TEGM)** [37], which use only one projection in each iteration:

$$\begin{cases} x_0 \in H, \\ y_n = P_C(x_n - \lambda A x_n), \\ x_{n+1} = y_n - \lambda (A y_n - A x_n), \quad \forall n \geq 0, \end{cases} \tag{6}$$

where $\lambda \in (0, \frac{1}{L})$.

Recently, the method **(TEGM)** for solving the problem **(VIP)** (1) has received great attention by many authors (see, for example, [31,35,38] and the references therein).

In this paper, motivated and inspired by the work of Censor et al. [5], Kraikaew and Saejung [17], first, we investigate the strong convergence for solving the problem **(VIP)** by our new algorithm which is a combination between the modified Tseng extragradient method and the viscosity method [20,25] for solving the problem **(VIP)** in Hilbert spaces. Second, we show that an advantage of the proposed algorithm is the computation of only two values of the inequality mapping and one projection onto the feasible set per one iteration, which distinguishes our method from most other projection-type methods for variational inequality problems with monotone mappings (see [5,12,17,23,38]). Finally, we give some numerical experiments for the performance of the proposed algorithm for variational inequality problems.

This paper is organized as follows: In Sect. 2, we recall some definitions and preliminary results for further use. Sect. 3 deals with analyzing the convergence of the proposed algorithms. Finally, in Sect. 4, we perform some numerical experiments to illustrate the behaviours of the proposed algorithms in comparison with other algorithms.

2 Preliminaries

Let H be a real Hilbert space and C be a nonempty closed convex subset of H.

- The weak convergence of $\{x_n\}$ to x is denoted by $x_n \rightharpoonup x$ as $n \to \infty$;
- The strong convergence of $\{x_n\}$ to x is written as $x_n \to x$ as $n \to \infty$.

For each $x, y \in H$, we have the following:

$$\|x+y\|^2 = \|x\|^2 + 2\langle x,y\rangle + \|y\|^2;$$
$$\|x+y\|^2 \le \|x\|^2 + 2\langle y, x+y\rangle; \tag{7}$$

For all point $x \in H$, there exists the unique nearest point in C, denoted by $P_C x$, such that

$$\|x - P_C x\| \le \|x - y\|, \quad \forall y \in C.$$

P_C is called the *metric projection* of H onto C. It is known that P_C is nonexpansive.

Lemma 2.1 ([13]) *Let C be a nonempty closed convex subset of a real Hilbert space H. For any $x \in H$ and $z \in C$, we have*

$$z = P_C x \iff \langle x - z, z - y\rangle \ge 0, \quad \forall y \in C.$$

Lemma 2.2 ([13]) *Let C be a closed convex subset in a real Hilbert space H and $x \in H$. Then*

(1) $\|P_C x - P_C y\|^2 \le \langle P_C x - P_C y, x - y\rangle$ *for all* $y \in C$;
(2) $\|P_C x - y\|^2 \le \|x - y\|^2 - \|x - P_C x\|^2$ *for all* $y \in C$.

For some more properties of the metric projection, refer to Section 3 in [13].

Lemma 2.3 ([15,28]) *Let $\{a_n\}$ be sequence of nonnegative real numbers, $\{\alpha_n\}$ be a sequence of real numbers in $(0,1)$ with $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\{b_n\}$ be a sequence of real numbers. Assume that*

$$a_{n+1} \leq (1-\alpha_n)a_n + \alpha_n b_n, \ \ \forall n \geq 1,$$

If $\limsup_{k\to\infty} b_{n_k} \leq 0$ *for every subsequence* $\{a_{n_k}\}$ *of* $\{a_n\}$ *satisfying* $\liminf_{k\to\infty}(a_{n_k+1} - a_{n_k}) \geq 0$ *then* $\lim_{n\to\infty} a_n = 0$.

Lemma 2.4 ([17]) *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $A : H \to H$ be a monotone and L-Lipschitz continuous mapping on C and let $S = P_C(I - \lambda A)$, where $\lambda > 0$. If $\{x_n\}$ is a sequence in H satisfying $x_n \rightharpoonup q$ and $x_n - Sx_n \to 0$, then $q \in VI(C,A) = Fix(S)$.*

3 Main results

Recently, some authors have studied the convergence of modified inertial Mann algorithms and inertial CQ-algorithms to fixed points of nonexpansive mappings ([10]) and the convergence of inertial projection and contraction algorithms to solutions of variational inequality problems in Hilbert spaces ([9]).

In this section, we construct Algorithm 3.1 by the modified Tseng extragradient via using the term $\alpha_n(x_n - x_{n-1})$, which is called the *inertia*, and it can be regarded as the procedure of speeding up the convergence properties (see, for example, [1,21,27]). Therefore, the following algorithm is different from the algorithms studied in [31,37,38].

Let $f : H \to H$ be a contraction mapping with contraction parameter $\kappa \in [0,1)$. Let $\lambda \in (0, \frac{1}{L})$, $\{\alpha_n\} \subset [0,\alpha)$ for some $\alpha > 0$ and $\{\beta_n\} \subset (0,1)$ satisfying the following conditions:

$$\lim_{n\to\infty} \beta_n = 0, \quad \sum_{n=1}^{\infty} \beta_n = \infty.$$

Now, we introduce the following algorithm:

Algorithm 3.1

Initialization: Let $x_0, x_1 \in H$ be arbitrary.

Iterative Steps: Calculate x_{n+1} as follows:

Step 1. Set $w_n = x_n + \alpha_n(x_n - x_{n-1})$ and compute

$$y_n = P_C(w_n - \lambda A w_n).$$

If $y_n = w_n$, then stop and y_n is a solution of the problem **(VIP)**.
Otherwise, go to **Step 2**.

Step 2. Comptue Compute

$$x_{n+1} = \beta_n f(x_n) + (1-\beta_n) z_n,$$

where $z_n = y_n - \lambda(Ay_n - Aw_n)$. Set $n := n+1$ and go to **Step 1**.

Now, we give our main results in this paper.

Theorem 3.1 *Assume that* (C1)*,* (C2)*,* (C3) *hold and the sequence* $\{\alpha_n\}$ *is chosen such that it satisfies the following condition:*

$$\lim_{n\to\infty} \frac{\alpha_n}{\beta_n} \|x_n - x_{n-1}\| = 0. \tag{8}$$

Then the sequence $\{x_n\}$ *generated by Algorithm 3.1 converges strongly to an element* $p \in VI(C,A)$*, where* $p = P_{VI(C,A)} \circ f(p)$*.*

Remark 3.1 In [29], Suantai et al. noted that the condition (8) is easily implemented in the numerical computation since the value of $\|x_n - x_{n-1}\|$ is known before choosing α_n. Indeed, the parameter α_n can be chosen such that

$$\alpha_n = \begin{cases} \min\{\frac{\tau_n}{\|x_n - x_{n-1}\|}, \dfrac{\alpha}{2}\} & \text{if } x_n \neq x_{n-1}, \\ \dfrac{\alpha}{2} & \text{if otherwise,} \end{cases}$$

where $\{\tau_n\}$ is a positive sequence such that $\tau_n = \circ(\beta_n)$.

Proof **Claim 1.** The sequence $\{x_n\}$ is bounded. Indeed, for $p = P_{VI(C,A)} \circ f(p)$, first, we show that

$$\|z_n - p\|^2 \le \|w_n - p\|^2 - (1-\lambda^2 L^2)\|y_n - w_n\|^2. \tag{9}$$

Now, we have

$$\begin{aligned} \|z_n - p\|^2 &= \|y_n - \lambda(Ay_n - Aw_n) - p\|^2 \\ &= \|y_n - p\|^2 + \lambda^2 \|Ay_n - Aw_n\|^2 - 2\lambda\langle y_n - p, Ay_n - Aw_n\rangle \\ &= \|w_n - p\|^2 + \|w_n - y_n\|^2 + 2\langle y_n - w_n, w_n - p\rangle + \lambda^2\|Ay_n - Aw_n\|^2 \\ &\quad - 2\lambda\langle y_n - p, Ay_n - Aw_n\rangle \\ &= \|w_n - p\|^2 + \|w_n - y_n\|^2 - 2\langle y_n - w_n, y_n - w_n\rangle + 2\langle y_n - w_n, y_n - p\rangle \\ &\quad + \lambda^2\|Ay_n - Aw_n\|^2 - 2\lambda\langle y_n - p, Ay_n - Aw_n\rangle \\ &= \|w_n - p\|^2 - \|w_n - y_n\|^2 + 2\langle y_n - w_n, y_n - p\rangle + \lambda^2\|Ay_n - Aw_n\|^2 \\ &\quad - 2\lambda\langle y_n - p, Ay_n - Aw_n\rangle. \end{aligned} \tag{10}$$

Since $y_n = P_C(w_n - \lambda Aw_n)$, we have

$$\langle y_n - w_n + \lambda Aw_n, y_n - p\rangle \le 0$$

or, equivalently,

$$\langle y_n - w_n, y_n - p\rangle \le -\lambda\langle Aw_n, y_n - p\rangle. \tag{11}$$

From (10) and (11), it follows that

$$\begin{aligned}
\|z_n - p\|^2 &\le \|w_n - p\|^2 - \|w_n - y_n\|^2 - 2\lambda \langle Aw_n, y_n - p\rangle + \lambda^2 \|Ay_n - Aw_n\|^2 \\
&\quad - 2\lambda \langle y_n - p, Ay_n - Aw_n\rangle \\
&= \|w_n - p\|^2 - \|w_n - y_n\|^2 + \lambda^2 \|Ay_n - Aw_n\|^2 - 2\lambda \langle y_n - p, Ay_n\rangle \\
&\le \|w_n - p\|^2 - \|w_n - y_n\|^2 + \lambda^2 L^2 \|y_n - w_n\|^2 - 2\lambda \langle y_n - p, Ay_n - Ap\rangle \\
&\quad - 2\lambda \langle y_n - p, Ap\rangle \\
&\le \|w_n - p\|^2 - (1 - \lambda^2 L^2)\|y_n - w_n\|^2.
\end{aligned}$$

Therefore, we have

$$\|z_n - p\| \le \|w_n - p\|. \tag{12}$$

From the definition of w_n, we get

$$\begin{aligned}
\|w_n - p\| &= \|x_n + \alpha_n (x_n - x_{n-1}) - p\| \\
&\le \|x_n - p\| + \alpha_n \|x_n - x_{n-1}\| \\
&= \|x_n - p\| + \beta_n \cdot \frac{\alpha_n}{\beta_n} \|x_n - x_{n-1}\|.
\end{aligned} \tag{13}$$

By the condition $\frac{\alpha_n}{\beta_n}\|x_n - x_{n-1}\| \to 0$, it follows that there exists a constant $M_1 > 0$ such that

$$\frac{\alpha_n}{\beta_n}\|x_n - x_{n-1}\| \le M_1, \ \ \forall n \ge 1. \tag{14}$$

Combining (12), (13) and (18), we obtain

$$\|z_n - p\| \le \|w_n - p\| \le \|x_n - p\| + \beta_n M_1. \tag{15}$$

From the definition of $\{x_n\}$, we get

$$\begin{aligned}
\|x_{n+1} - p\| &= \|\beta_n f(x_n) + (1 - \beta_n) z_n - p\| \\
&= \|\beta_n (f(x_n) - p) + (1 - \beta_n)(z_n - p)\| \\
&\le \beta_n \|f(x_n) - p\| + (1 - \beta_n)\|z_n - p\| \\
&\le \beta_n \|f(x_n) - f(p)\| + \beta_n \|f(p) - p\| + (1 - \beta_n)\|z_n - p\| \\
&\le \beta_n \kappa \|x_n - p\| + \beta_n \|f(p) - p\| + (1 - \beta_n)\|z_n - p\|.
\end{aligned} \tag{16}$$

Substituting (15) into (16), we obtain

$$\begin{aligned}
\|x_{n+1} - p\| &\le (1 - (1 - \kappa)\beta_n)\|x_n - p\| + \beta_n M_1 + \beta_n \|f(p) - p\|. \\
&= (1 - (1 - \kappa)\beta_n)\|x_n - p\| + (1 - \kappa)\beta_n \frac{M_1 + \|f(p) - p\|}{1 - \kappa} \\
&\le \max\left\{\|x_n - p\|, \frac{M_1 + \|f(p) - p\|}{1 - \kappa}\right\} \\
&\le \cdots \\
&\le \max\left\{\|x_0 - p\|, \frac{M_1 + \|f(p) - p\|}{1 - \kappa}\right\}.
\end{aligned}$$

This implies $\{x_n\}$ is bounded. We also get $\{z_n\}, \{f(x_n)\}, \{w_n\}$ are bounded.

Claim 2.

$$(1-\beta_n)(1-\lambda^2L^2)\|y_n-w_n\|^2 \le \|x_n-q\|^2-\|x_{n+1}-q\|^2+\beta_nM_4 \tag{17}$$

for some $M_4>0$. Indeed, we get

$$\begin{aligned}\|x_{n+1}-p\|^2 &\le \beta_n\|f(x_n)-p\|^2+(1-\beta_n)\|z_n-p\|^2\\ &\le \beta_n(\|f(x_n)-f(p)\|+\|f(p)-p\|)^2+\|z_n-p\|^2\\ &\le \beta_n(\kappa\|x_n-p\|+\|f(p)-p\|)^2+(1-\beta_n)\|z_n-p\|^2\\ &\le \beta_n(\|x_n-p\|+\|f(p)-p\|)^2+(1-\beta_n)\|z_n-p\|^2\\ &= \beta_n\|x_n-p\|^2+\beta_n(2\|x_n-p\|\cdot\|f(p)-p\|\\ &\quad+\|f(p)-p\|^2)+(1-\beta_n)\|z_n-p\|^2\\ &\le \beta_n\|x_n-p\|^2+(1-\beta_n)\|z_n-p\|^2+\beta_nM_2 \end{aligned}\tag{18}$$

for some $M_2>0$. Substituting (9) into (18), we get

$$\begin{aligned}\|x_{n+1}-p\|^2 \le \beta_n\|x_n-p\|^2&+(1-\beta_n)\|w_n-p\|^2\\ &-(1-\beta_n)(1-\lambda^2L^2)\|y_n-w_n\|^2+\beta_nM_2,\end{aligned}\tag{19}$$

which implies from (15) that

$$\begin{aligned}\|w_n-p\|^2 &\le (\|x_n-p\|+\beta_nM_1)^2\\ &=\|x_n-p\|^2+\beta_n(2M_1\|x_n-p\|+\beta_nM_1^2)\\ &\le \|x_n-p\|^2+\beta_nM_3,\end{aligned}\tag{20}$$

for some $M_3>0$. Combining (19) and (20), we obtain

$$\begin{aligned}\|x_{n+1}-p\|^2 &\le \beta_n\|x_n-p\|^2+(1-\beta_n)\|x_n-p\|^2+\beta_nM_3\\ &\quad-(1-\beta_n)(1-\lambda^2L^2)\|y_n-w_n\|^2+\beta_nM_2\\ &=\|x_n-p\|^2+\beta_nM_3-(1-\beta_n)(1-\lambda^2L^2)\|y_n-w_n\|^2+\beta_nM_2.\end{aligned}$$

This implies that

$$(1-\beta_n)(1-\lambda^2L^2)\|y_n-w_n\|^2 \le \|x_n-q\|^2-\|x_{n+1}-q\|^2+\beta_nM_4,$$

where $M_4:=M_2+M_3$.

Claim 3.

$$\begin{aligned}\|x_{n+1}-p\|^2 &\le (1-(1-\kappa)\beta_n)\|x_n-p\|^2\\ &\quad+(1-\kappa)\beta_n\cdot\Big[\frac{2}{1-\kappa}\langle f(p)-p,x_{n+1}-p\rangle+\frac{3M}{1-\kappa}\cdot\frac{\alpha_n}{\beta_n}\cdot\|x_n-x_{n-1}\|\Big]\end{aligned}$$

for some $M>0$. Indeed, we have

$$\begin{aligned}\|w_n-p\|^2 &= \|x_n+\alpha_n(x_n-x_{n-1})-p\|^2\\ &=\|x_n-p\|^2+2\alpha_n\langle x_n-p,x_n-x_{n-1}\rangle+\alpha_n^2\|x_n-x_{n-1}\|^2\\ &\le \|x_n-p\|^2+2\alpha_n\|x_n-p\|\|x_n-x_{n-1}\|+\alpha_n^2\|x_n-x_{n-1}\|^2.\end{aligned}\tag{21}$$

Using (7), we have

$$\begin{aligned}\|x_{n+1}-p\|^2 &= \|\beta_n f(x_n)+(1-\beta_n)z_n-p\|^2\\ &= \|\beta_n(f(x_n)-f(p))+(1-\beta_n)(z_n-p)+\beta_n(f(p)-p)\|^2\\ &\le \|\beta_n(f(x_n)-f(p))+(1-\beta_n)(z_n-p)\|^2+2\beta_n\langle f(p)-p,x_{n+1}-p\rangle\\ &\le \beta_n\|f(x_n)-f(p)\|^2+(1-\beta_n)\|z_n-p\|^2+2\beta_n\langle f(p)-p,x_{n+1}-p\rangle\\ &\le \beta_n\kappa^2\|x_n-p\|^2+(1-\beta_n)\|z_n-p\|^2+2\beta_n\langle f(p)-p,x_{n+1}-p\rangle\\ &\le \beta_n\kappa\|x_n-p\|^2+(1-\beta_n)\|z_n-p\|^2+2\beta_n\langle f(p)-p,x_{n+1}-p\rangle\\ &\le \beta_n\kappa\|x_n-p\|^2+(1-\beta_n)\|w_n-p\|^2+2\beta_n\langle f(p)-p,x_{n+1}-p\rangle. \end{aligned} \tag{22}$$

Substituting (21) into (22), we have

$$\begin{aligned}&\|x_{n+1}-p\|^2\\ &\le (1-(1-\kappa)\beta_n)\|x_n-p\|^2+2\alpha_n\|x_n-x^*\|\|x_n-x_{n-1}\|+\alpha_n^2\|x_n-x_{n-1}\|^2\\ &\quad+2\beta_n\langle f(p)-p,x_{n+1}-p\rangle\\ &= (1-(1-\kappa)\beta_n)\|x_n-p\|^2+(1-\kappa)\beta_n\cdot\frac{2}{1-\kappa}\langle f(p)-p,x_{n+1}-p\rangle\\ &\quad+\alpha_n\|x_n-x_{n-1}\|(2\|x_n-x^*\|+\alpha_n\|x_n-x_{n-1}\|)\\ &\le (1-(1-\kappa)\beta_n)\|x_n-p\|^2+(1-\kappa)\beta_n\cdot\frac{2}{1-\kappa}\langle f(p)-p,x_{n+1}-p\rangle\\ &\quad+\alpha_n\|x_n-x_{n-1}\|(2\|x_n-x^*\|+\alpha\|x_n-x_{n-1}\|)\\ &\le (1-(1-\kappa)\beta_n)\|x_n-p\|^2\\ &\quad+(1-\kappa)\beta_n\cdot\frac{2}{1-\kappa}\langle f(p)-p,x_{n+1}-p\rangle+3M\alpha_n\|x_n-x_{n-1}\|\\ &\le (1-(1-\kappa)\beta_n)\|x_n-p\|^2\\ &\quad+(1-\kappa)\beta_n\cdot\Big[\frac{2}{1-\kappa}\langle f(p)-p,x_{n+1}-p\rangle+\frac{3M}{1-\kappa}\cdot\frac{\alpha_n}{\beta_n}\cdot\|x_n-x_{n-1}\|\Big],\end{aligned}$$

where $M:=\sup_{n\in\mathbb{N}}\{\|x_n-x^*\|,\theta\|x_n-x_{n-1}\|\}>0$.

Claim 4. $\{\|x_n-p\|^2\}$ converges to zero. Indeed, by Lemma 2.3 it suffices to show that $\limsup_{k\to\infty}\langle f(p)-p,x_{n_k+1}-p\rangle\le 0$ for every subsequence $\{\|x_{n_k}-p\|\}$ of $\{\|x_n-p\|\}$ satisfying

$$\liminf_{k\to\infty}(\|x_{n_k+1}-p\|-\|x_{n_k}-p\|)\ge 0.$$

For this, suppose that $\{\|x_{n_k}-p\|\}$ is a subsequence of $\{\|x_n-p\|\}$ such that $\liminf_{k\to\infty}(\|x_{n_k+1}-p\|-\|x_{n_k}-p\|)\ge 0$. Then

$$\liminf_{k\to\infty}(\|x_{n_k+1}-p\|^2-\|x_{n_k}-p\|^2)=\liminf_{k\to\infty}[(\|x_{n_k+1}-p\|-\|x_{n_k}-p\|)(\|x_{n_k+1}-p\|+\|x_{n_k}-p\|)]\ge 0.$$

By Claim 2 we obtian

$$\begin{aligned}\limsup_{k\to\infty}(1-\beta_{n_k})(1-\lambda^2L^2)\|y_{n_k}-w_{n_k}\|^2 &\le \limsup_{k\to\infty}[\|x_{n_k}-p\|^2-\|x_{n_k+1}-q\|^2+\beta_{n_k}M_4]\\ &\le \limsup_{k\to\infty}[\|x_{n_k}-p\|^2-\|x_{n_k}-p\|^2]+\limsup_{k\to\infty}\beta_{n_k}M_4\\ &= -\liminf_{k\to\infty}[\|x_{n_k+1}-q\|^2-\|x_{n_k}-p\|^2]\\ &\le 0.\end{aligned}$$

This implies that

$$\lim_{k\to\infty} \|y_{n_k} - w_{n_k}\| = 0. \tag{23}$$

Now, we show that

$$\|x_{n_k+1} - x_{n_k}\| \to 0 \text{ as } n \to \infty. \tag{24}$$

Indeed, from (23), it follows that

$$\begin{aligned}\|z_{n_k} - w_{n_k}\| &= \|y_{n_k} - \lambda(Ay_{n_k} - Aw_{n_k}) - w_{n_k}\| \\ &\le \|y_{n_k} - w_{n_k}\| + \lambda\|Ay_{n_k} - Aw_{n_k}\| \\ &\le (1+\lambda L)\|y_{n_k} - w_{n_k}\| \to 0. \end{aligned}\tag{25}$$

Moreover, we have

$$\|x_{n_k+1} - z_{n_k}\| = \beta_{n_k}\|z_{n_k} - f(x_{n_k})\| \to 0, \tag{26}$$

and

$$\|x_{n_k} - w_{n_k}\| = \alpha_{n_k}\|x_{n_k} - x_{n_k-1}\| = \beta_{n_k}.\frac{\alpha_{n_k}}{\beta_{n_k}}\|x_{n_k} - x_{n_k-1}\| \to 0. \tag{27}$$

From (25), (26) and (27), we get

$$\|x_{n_k+1} - x_{n_k}\| \le \|x_{n_k+1} - z_{n_k}\| + \|z_{n_k} - w_{n_k}\| + \|w_{n_k} - x_{n_k}\| \to 0.$$

Since the sequence $\{x_{n_k}\}$ is bounded, it follows that there exists a subsequence $\{x_{n_{k_j}}\}$ of $\{x_{n_k}\}$, which converges weakly to some $z \in H$, such that

$$\limsup_{k\to\infty}\langle f(p) - p, x_{n_k} - p\rangle = \lim_{j\to\infty}\langle f(p) - p, x_{n_{k_j}} - p\rangle = \langle f(p) - p, z - p\rangle. \tag{28}$$

From (23) and Lemma 2.4, we have $z \in VI(C,A)$ and, from (28) and the definition of $p = P_{VI(C,A)} \circ f(p)$, we have

$$\limsup_{k\to\infty}\langle f(p) - p, x_{n_k} - p\rangle = \langle f(p) - p, z - p\rangle \le 0. \tag{29}$$

Combining (24) and (29), we have

$$\begin{aligned}\limsup_{k\to\infty}\langle f(p) - p, x_{n_k+1} - p\rangle &\le \limsup_{k\to\infty}\langle f(p) - p, x_{n_k} - p\rangle \\ &= \langle f(p) - p, z - p\rangle \\ &\le 0. \end{aligned}\tag{30}$$

Hence, by (30), Claim 3 and Lemma 2.3, we have $\lim_{n\to\infty}\|x_n - p\| = 0$. That is the desired result.

4 Numerical illustrations

In this section, we provide two numerical examples to test the proposed algorithm. All the codes were written in Matlab (R2015a) and run on PC with Intel(R) Core(TM) i3-370M Processor 2.40 GHz.

Now, we apply Algorithm 3.1 to solve the variational inequality problem **(VIP)** and compare numerical results with other algorithms. In the numerical results reported in the following tables, 'Iter.' and 'Sec.' denote the number of iterations and the cpu time in seconds, respectively.

Example 1 Suppose that $H = L^2([0,1])$ with the inner product

$$\langle x,y\rangle := \int_0^1 x(t)y(t)dt, \ \ \forall x,y \in H$$

and the induced norm

$$\|x\| := \left(\int_0^1 |x(t)|^2 dt\right)^{\frac{1}{2}}, \ \ \forall x \in H.$$

Let $C := \{x \in H : \|x\| \leq 1\}$ be the unit ball and define an operator $A : C \to H$ by

$$(Ax)(t) = \max\{0, x(t)\}.$$

It is easy to see that A is 1-Lipschitz continuous and monotone on C. With these given C and A, the set of solutions to the variational inequality problem **(VIP)** is given by

$$\Gamma = \{0\} \neq \emptyset.$$

It is known that

$$P_C(x) = \begin{cases} \frac{x}{\|x\|_{L^2}}, & \text{if } \|x\|_{L^2} > 1, \\ x, & \text{if } \|x\|_{L^2} \leq 1. \end{cases}$$

Now, we apply Algorithm 3.1 **(iTEM)**, Maingé's algorithm [22] (**M** and Kraikaew and Saejung's algorithm [17] (**KR**) to solve the variational inequality problem **(VIP)**.

We use the following:

(i) the same parameter $\lambda = 0.5$;
(ii) the stopping rule $\|y_n - w_n\| < 10^{-3}$ for Algorithm 3.1 and $\|y_n - x_n\| < 10^{-3}$ for Maingé's algorithm and Kraikaew and Saejung's algorithm;
(iii) the same starting point x_0.

Moreover, with respect to Algorithm 3.1, we take $f(x) = \frac{x}{8}$, $\beta_n = \frac{1}{n+1}$, $\theta = 0.6$ and

$$\alpha_n = \begin{cases} \min\left\{\theta, \dfrac{\beta_n^2}{\|x_n - x_{n-1}\|}\right\} & \text{if } x_n \neq x_{n-1}, \\ \theta & \text{otherwise.} \end{cases} \tag{31}$$

We also choose $\alpha_n = \frac{1}{n+1}$ for Maingé's algorithm **(M)** and Kraikaew and Saejung's algorithm **(KR)**. We now make comparison of three algorithms with different x_0 and report the results in Table 4.1.

	$x_0(t) = t/200$		$x_0(t) = \frac{1}{252}(t^2 - e^{-t})$		$x_0(t) = \frac{1}{525}(\sin(-3t) + \cos(-10t))$	
	Sec.	Iter.	Sec.	Iter.	Sec.	Iter.
Algorithm KR	0.076122	183	0.020808	49	0.011425	28
Algorithm M	0.025144	183	0.0062326	49	0.0035868	28
iTEM	0.0034307	19	0.0024726	15	0.0023558	14

Table 4.1: Comparison of three algorithms in Example 1

The convergence behaviour of algorithms with different starting point is given in Figures 1-3. In these figures, the value of errors $\|y_n - w_n\|$ (our **Algorithm 3.1**) and $\|y_n - x_n\|$ (Algorithm **(KR)** and Algorithm **(M)** is represented by the y-axis, number of iterations is represented by the x-axis.

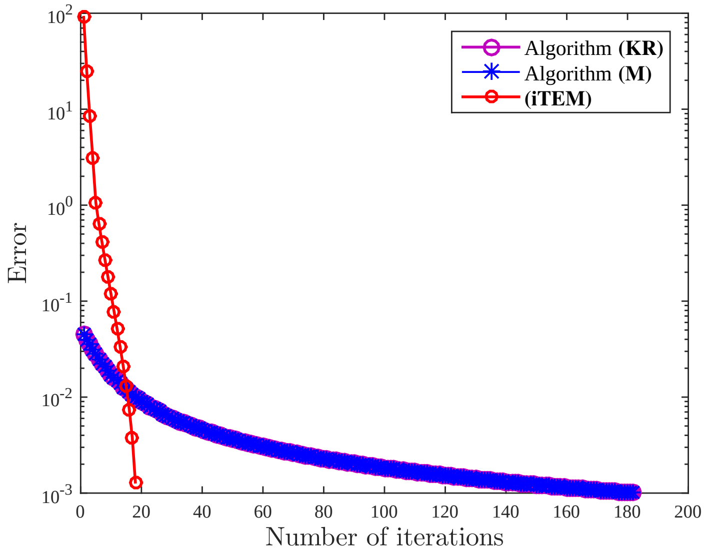

Fig. 1: Comparison of three algorithms in Example 1 with $x_0 = t/200$.

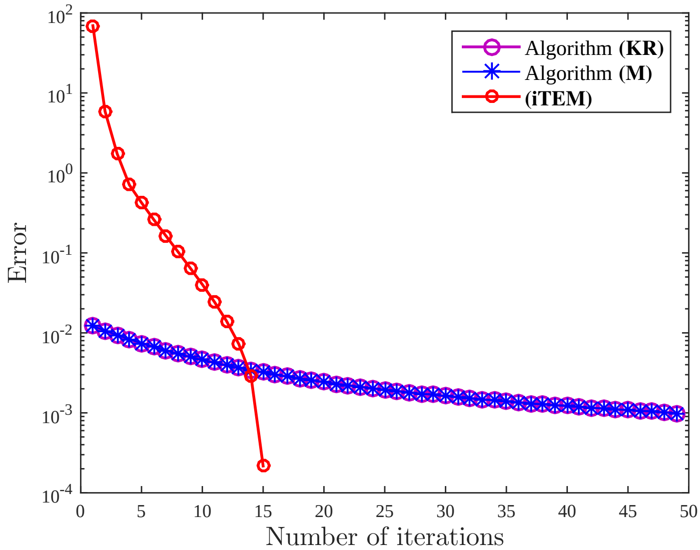

Fig. 2: Comparison of three algorithms in Example 1 with $x_0 = \frac{1}{252}(t^2 - e^{-t})$.

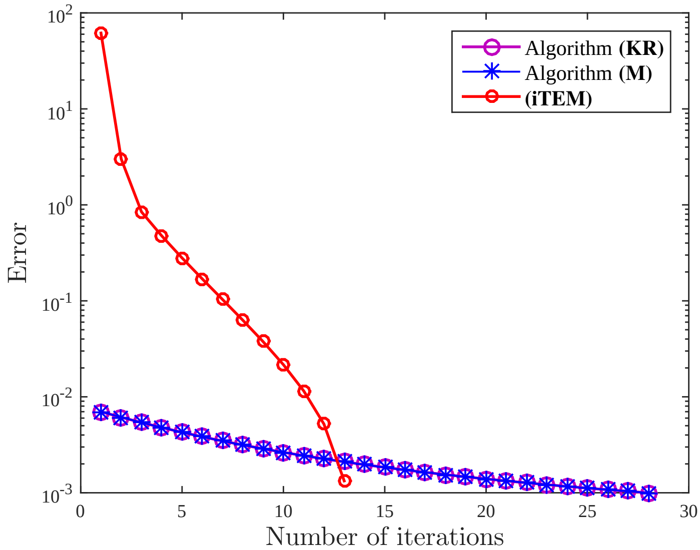

Fig. 3: Comparison of three algorithms in Example 1 with $x_0 = \frac{1}{525}(\sin(-3t) + \cos(-10t))$.

Example 2 Consider the linear operator $A : \mathbb{R}^m \to \mathbb{R}^m$ defined by $A(x) = Mx + q$, which is taken from [14] and has been considered by many authors for numerical experiments, see, for example, [24, 31], where

$$M = BB^T + S + D,$$

B is an $m \times m$ matrix, S is an $m \times m$ skew-symmetric matrix, D is an $m \times m$ diagonal matrix, whose diagonal entries are nonnegative (so M is positive definite), q is a vector in $\mathbb{R}^m$. It is clear that A is monotone and Lipschitz continuous with the Lipschitz constant $L = ||M||$. For $q = 0$, the unique solution of the corresponding variational inequality is $\{0\}$. We will compare our Algorithm 3.1 **(iTEM)** with the standard algorithm (Algorithm 3.1 with $\alpha_k = 0$, shortly, **(TEM)**. The starting point is $x_0 = (1, 1, \cdots, 1) \in \mathbb{R}^m$. All entries of the matrices B, S, D are generated randomly (matrices of normally distributed random numbers).

The feasible set, control parameters and stopping rulues are choosen as in Example 1 except $\beta_n = \frac{1}{n+2}$. The results are described in Table 4.2 and Figures 4–6.

	$m = 10$		$m = 40$		$m = 80$	
	Sec.	Iter.	Sec.	Iter.	Sec.	Iter.
iTEM	0.4056	56	0.4531	64	1.3281	75
TEM	2.3868	396	4.0313	618	12.1563	710

Table 4.2: Comparison of two algorithms with different m

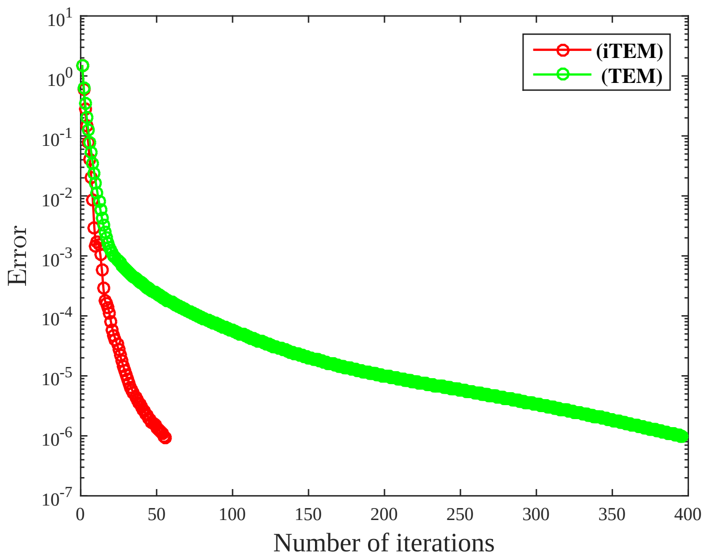

Fig. 4: Comparison of two algorithms in Example 2 with $m = 10$.

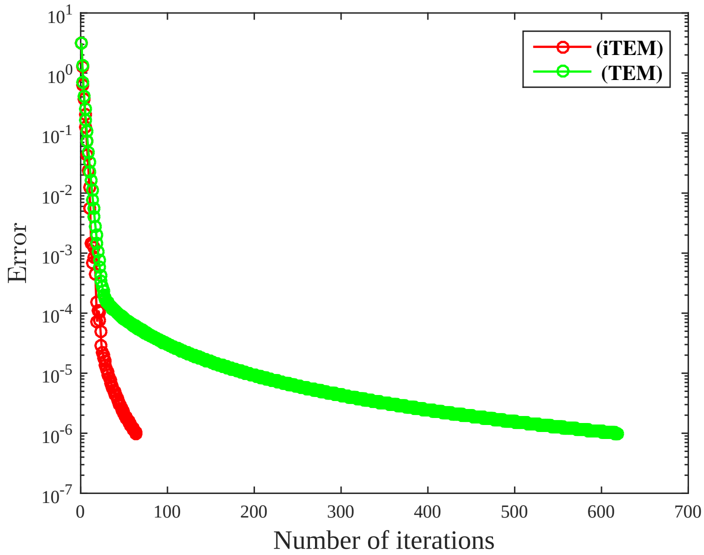

Fig. 5: Comparison of two algorithms in Example 2 with $m = 40$.

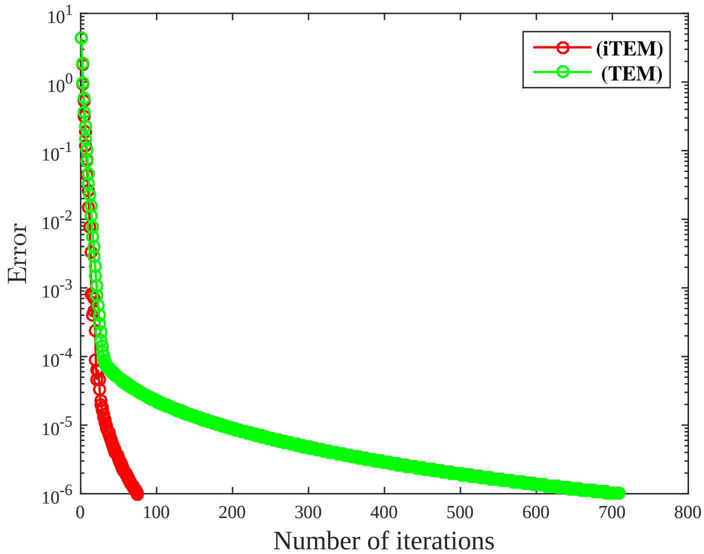

Fig. 6: Comparison of two algorithms in Example 2 with $m = 80$.

5 Conclusions

The paper has proposed a new method for solving monotone and Lipschitz VIPs in Hilbert spaces. Under some suitable conditions imposed on parameters, we have proved the strong convergence of the algorithm. The efficiency of the proposed algorithms has also been illustrated by several numerical experiments.

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