

GENERALIZED COHEN-MACAULAYNESS AND NON-COHEN-MACAULAY LOCUS OF CANONICAL MODULES

TRAN NGUYEN AN

College of Education, Thai Nguyen University
Thai Nguyen, Vietnam
e-mail: antrannguyen@gmail.com

LE THANH NHAN

College of Science, Thai Nguyen University
Thai Nguyen, Vietnam
e-mail: nhanlt2014@gmail.com

LUU PHUONG THAO

College of Education, Thai Nguyen University
Thai Nguyen, Vietnam
e-mail: thaoktsp@gmail.com

Abstract ¹. Let $(R, \mathfrak{m})$ be a Noetherian local ring which is a quotient of a Gorenstein local ring. Let M be a finitely generated R -module. Denote by K_M the canonical module of M . In this paper, we study the generalized Cohen-Macaulayness and the non-Cohen-Macaulay locus of K_M . Firstly we introduce the notion of canonical system of parameters of M in order to characterize the generalized Cohen-Macaulayness of K_M . We give two other parametric characterizations for K_M to be generalized Cohen-Macaulay. Then we present the relation between the non-Cohen-Macaulay locus of K_M and that of M .

1 Introduction

The depth and the Cohen-Macaulayness of canonical modules have attracted the interest of a number of researchers, see [A], [AG], [Sch1], [Nh], [BN]. Aoyama and Goto [AG] proved that if R is a Noetherian local with the total quotient ring $Q(R)$ such that R is unmixed and R admits the canonical module K_R , then K_R is a Cohen-Macaulay R -module if and only if there exists a Cohen-Macaulay intermediate ring B between R and $Q(R)$ such that B is a finitely generated R -module with $\dim_R(B/R) \leq \dim R - 2$ and $\dim B_{\mathfrak{n}} = \dim R$ for any maximal ideal $\mathfrak{n}$ of B . However, the fact is not valid any more whenever $\dim_R(B/R) = \dim R - 1$, see Example 2.5.

¹*Key words and phrases:* Generalized Cohen-Macaulay canonical module, canonical sequence, strict f-sequence.

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Let $(R, \mathfrak{m})$ be a Noetherian local ring, let M be a finitely generated R -module with $\dim M = d$. For each system of parameters (s.o.p. for short) $(x_1, \dots, x_d)$ of M , set

$$I(x_1, \dots, x_d; M) = \ell_R(M/(x_1, \dots, x_d)M) - e(x_1, \dots, x_d; M).$$

It is well-known that M is Cohen-Macaulay if and only if $I(x_1, \dots, x_d; M) = 0$ for some (for all) s.o.p. $(x_1, \dots, x_d)$ of M . A similar parametric characterization for the Cohen-Macaulayness of canonical module was given in [Nh] and [BN] as follows. Suppose that R is a quotient of a Gorenstein local ring. Denote by K_M the canonical module of M . For an Artinian R -module A , denote by $\text{Rl}(A) := \ell_R(A/\mathfrak{m}^s A)$ the residual length of A defined by Sharp and Hamieh [SH], where $s > 0$ is an integer such that $\mathfrak{m}^s A = \mathfrak{m}^n A$ for all $n \geq s$. Then K_M is Cohen-Macaulay if and only if $\text{Rl}(H_{\mathfrak{m}}^2(M/(x_1, \dots, x_{d-3})M)) = 0$ for some (for all) strict f-sequence $(x_1, \dots, x_d)$ of M . Here, the notion of strict f-sequence was introduced in [CMN], and if $(x_1, \dots, x_d)$ is a strict f-sequence of M , then it is a s.o.p. of M .

Set $I(M) := \sup I(x_1, \dots, x_d; M)$, where the supremum runs over all s.o.p $(x_1, \dots, x_d)$ of M . We say that M is *generalized Cohen-Macaulay* if $I(M) < \infty$, see [CST].

Theorem 1.1. (See [CST], [Tr]). *The following statements are equivalent:*

- (a) M is generalized Cohen-Macaulay;
- (b) There exists a s.o.p. $(x_1, \dots, x_d)$ of M such that $\sup_{n_1, \dots, n_d \in \mathbb{N}} I(x_1^{n_1}, \dots, x_d^{n_d}; M) < \infty$;
- (c) M has a standard s.o.p. $(x_1, \dots, x_d)$, i.e. $I(x_1, \dots, x_d; M) = I(x_1^2, \dots, x_d^2; M)$.

In this paper, firstly we establish an analogue for the canonical modules of the parametric characterizations in Theorem 1.1 for generalized Cohen-Macaulay modules, where the role of the number $\text{Rl}(H_{\mathfrak{m}}^2(M/(x_1, \dots, x_{d-3})M))$ in the study of K_M is as useful as that of the number $I(x_1, \dots, x_d; M)$ in the study of M , for strict f-sequences $(x_1, \dots, x_d)$ of M . We introduce the notion of canonical system of parameters (canonical s.o.p. for short) as follows.

Definition 1.2. A strict f-sequence $\underline{x} = (x_1, \dots, x_d)$ is said to be a *canonical s.o.p.* of M if

$$\text{Rl}(H_{\mathfrak{m}}^2(M/(x_1, \dots, x_{d-3})M)) = \text{Rl}(H_{\mathfrak{m}}^2(M/(x_1^2, \dots, x_{d-3}^2)M)).$$

If $\underline{x}$ is at the same time an unconditioned strict f-sequence and a canonical s.o.p. of M , then $\underline{x}$ is said to be an *unconditioned canonical s.o.p.* of M .

The following theorem is the first main result of this paper.

Theorem 1.3. *Suppose that R is a quotient of a Gorenstein local ring. The following four statements are equivalent:*

- (a) K_M is generalized Cohen-Macaulay.
- (b) $c_M := \sup_{\underline{x}} \text{Rl}(H_{\mathfrak{m}}^2(M/(x_1, \dots, x_{d-3})M)) < \infty$ where $\underline{x} = (x_1, \dots, x_d)$ runs over all strict f-sequences of M .

(c) *There exists a strict f-sequence $(x_1, \dots, x_d)$ of M such that*

$$\sup_{n_1, \dots, n_{d-3} \in \mathbb{N}} \text{Rl} \left(H_{\mathfrak{m}}^2(M/(x_1^{n_1}, \dots, x_{d-3}^{n_{d-3}})M) \right) < \infty.$$

(d) *There is an unconditioned canonical s.o.p. of M .*

Futhermore, if $(x_1, \dots, x_d)$ is an unconditioned canonical s.o.p. of M , then

$$\text{Rl} \left(H_{\mathfrak{m}}^2(M/(x_1, \dots, x_{d-3})M) \right) = c_M = \sum_{i=0}^{d-3} \binom{d-3}{i} \ell(H_{\mathfrak{m}}^{i+2}(K_M)).$$

It should be mentioned that the statements (b) and (c) of Theorem 1.3 improve the main result of [LN].

Secondly, Y. Aoyama [A, Theorem 1] studied the relation between the depth of K_R and that of R in case where R is not Cohen-Macaulay. He proved that for given integers $0 \leq r < n$ and $2 \leq s \leq n$, there exists a complete local ring R such that $\dim R = n$, $\text{depth } R = r$ and $\text{depth } K_R = s$. This is the motivation for us to discuss about the relation between the non Cohen-Macaulay locus of K_M and that of M .

Denote by $\text{nCM}(M)$ the non-Cohen-Macaulay locus of M . If R is a quotient of a Gorenstein local ring, then $\text{nCM}(M)$ is closed under Zariski topology and $\dim \text{nCM}(M) \leq d - 1$, see [C]. Moreover, if M is unmixed, then $\dim \text{nCM}(M) \leq d - 2$.

The following theorem is the second main result of this paper.

Theorem 1.4. *Suppose that R is a quotient of a Gorenstein local ring. The following statements are true.*

- (a) $\dim \text{nCM}(K_M) \leq \min\{d - 3, \dim \text{nCM}(M)\}$.
- (b) *For given integers $-1 \leq s \leq d - 3$ and $s \leq r \leq d - 2$, there exists a complete unmixed Noetherian local ring $(R, \mathfrak{m})$ such that $\dim \text{nCM}(R) = r$ and $\dim \text{nCM}(K_R) = s$.*

In the next section, we present some preliminaries that will be used in the sequel. Section 3 and Section 4 are devoted to prove the main results of this paper (Theorems 1.3, 1.4).

2 Preliminaries

Throughout this paper, let $(R, \mathfrak{m})$ be a Noetherian local ring which is a quotient of an n -dimensional local Gorenstein ring $(R', \mathfrak{m}')$. Let M be a finitely generated R -module with $\dim M = d$. For each integer $i \geq 0$, let $K_M^i := \text{Ext}_{R'}^{n-i}(M, R')$ denote the i -th *deficiency module* of M . Then K_M^i is a finitely generated R -module and the local duality (see [BS, 11.2.6]) gives the isomorphism $H_{\mathfrak{m}}^i(M) \cong \text{Hom}_R(K_M^i, E(R/\mathfrak{m}))$, where $E(R/\mathfrak{m})$ is the injective envelope of $R/\mathfrak{m}$. Let K_M be the *canonical module* K_M^d of M . For an Artinian

R -module A , let $\text{Rl}(A) := \ell_R(A/\mathfrak{m}^s A)$ be the *residual length* of A defined by Sharp-Hamieh [SH], where $s > 0$ is an integer such that $\mathfrak{m}^n A = \mathfrak{m}^s A$ for all $n \geq s$.

The notion of filter regular sequence (f-sequence for short) introduced in [CST] can be considered as a generalization of the known concept of regular sequence. An element $x \in \mathfrak{m}$ is said to be a *filter regular element* (f-element for short) of M if $x \notin \mathfrak{p}$ for all $\mathfrak{p} \in \text{Ass}_R M \setminus \{\mathfrak{m}\}$. A sequence $(x_1, \dots, x_t)$ of elements in $\mathfrak{m}$ is said to be an f-sequence of M if x_i is an f-element of $M/(x_1, \dots, x_{i-1})M$ for all $i \leq t$.

Remark 2.1. *An element $x \in \mathfrak{m}$ is an f-element of M if and only if $\ell_R(0 :_M x) < \infty$. Moreover, each f-sequence of length d is a s.o.p. of M .*

A special kind of f-sequences is the class of strict f-sequences introduced in [CMN]. In the original definition of strict f-sequence, the set of attached primes $\text{Att}_R H_{\mathfrak{m}}^i(M)$ defined by I. G. Macdonald [Mac] was used. However, we note that $\text{Att}_R H_{\mathfrak{m}}^i(M) = \text{Ass}_R K_M^i$ by [S, Theorem 2.3], therefore we can recall the notion of strict f-sequence as follows.

Definition 2.2. An element $x \in \mathfrak{m}$ is said to be a *strict f-element* of M if $x \notin \mathfrak{p}$ for all $\mathfrak{p} \in \left(\bigcup_{i=1}^d \text{Ass}_R K_M^i \right) \setminus \{\mathfrak{m}\}$. A sequence $(x_1, \dots, x_t)$ of elements in $\mathfrak{m}$ is said to be a *strict f-sequence* of M if x_{j+1} is a strict f-element of $M/(x_1, \dots, x_j)M$ for all $j = 0, \dots, t-1$. A sequence $(x_1, \dots, x_t)$ of elements in $\mathfrak{m}$ is said to be an *unconditioned strict f-sequence* of M if it is a strict f-sequence in any order.

Note that $\text{Ass}_R M \subseteq \bigcup_{i=0}^d \text{Ass}_R K_M^i$, see [Sch2, Proposition 2.3(c)]. Hence, each strict f-sequence is an f-sequence of M . In particular, if $x \in \mathfrak{m}$ is a strict f-element of M , then $\ell_R(0 :_M x) < \infty$. Moreover, if $(x_1, \dots, x_d)$ is a strict f-sequence, then it is a s.o.p. of M .

Here are some properties of strict f-sequence that we need in the proof of the main results.

Lemma 2.3. ([CMN, Lemmas 3.4, 4.2], [GN, Theorem 3.5])

- (a) *A sequence $(x_1, \dots, x_t)$ of elements in $\mathfrak{m}$ is a strict f-sequence of M if and only if it is an f-sequence of K_M^i for all integers $i \geq 0$.*
- (b) *If $(x_1, \dots, x_t) \in \mathfrak{m}$ is a strict f-sequence of M , then so is $(x_1^{n_1}, \dots, x_t^{n_t})$ for all positive integers $n_1, \dots, n_t$.*
- (c) *For each integer $t > 0$, there exists an unconditioned strict f-sequence of M of length t .*

Lemma 2.4. ([LN, Lemmas 2.5, 2.7]) *Let $x \in \mathfrak{m}$ be a strict f-element of M . The following statements are true.*

- (a) *For each integer $i \geq 0$, there exists an integer n_0 such that for all $n \geq n_0$ we have*

$$\text{Rl}(H_{\mathfrak{m}}^i(M)) = \ell_R(H_{\mathfrak{m}}^0(K_M^i)) = \ell_R(0 :_{K_M^i} x^n).$$

(b) For each integer $i \geq 1$, there is an exact sequence

$$0 \rightarrow K_M^{i+1}/xK_M^{i+1} \rightarrow K_{M/xM}^i \rightarrow (0 :_{K_M^i} x) \rightarrow 0.$$

In particular, $H_{\mathfrak{m}}^i(K_M/xK_M) \cong H_{\mathfrak{m}}^i(K_{M/xM})$ for any $i \geq 2$.

Next, we discuss about the Cohen-Macaulayness and generalized Cohen-Macaulayness of the canonical module. Following P. Schenzel [Sch2, Definition 5.1], R is said to have a *birational Macaulayfication* if there is an intermediate ring B between R and $Q(R)$ such that B is a finitely generated Cohen-Macaulay R -module. As we mentioned in the introduction, Aoyama and Goto [AG] proved that if R is unmixed, then K_R is Cohen-Macaulay if and only if there exists a birational Macaulayfication B of R such that $\dim_R(B/R) \leq \dim R - 2$. When this is the case, B is uniquely determined and $B \cong \text{End}_R(K_R)$ as an R -algebra. Note that the condition $\dim_R(B/R) \leq \dim R - 2$ can not be removed. The following example given by S. Goto shows that the result does not valid any more if $\dim_R(B/R) = \dim R - 1$.

Example 2.5. Let $A = F[X, Y]$ be the polynomial ring over an infinite field F and $J = (X^3, V)(X^3, XV, Y^3)$, where $V = X^2 + XY + Y^2$. Let $\mathfrak{M} = (X, Y)$. Then $\sqrt{J} = \mathfrak{M}$. We set $I = JA_{\mathfrak{M}}$. Then the Rees algebra $\mathcal{R} = \mathcal{R}(I)$ of I is a Buchsbaum ring with depth $\mathcal{R}(I) = 2$. Since $A_{\mathfrak{M}}$ is a regular local ring of dimension 2, $\overline{\mathcal{R}} = \mathcal{R}(\overline{I})$ is a Cohen-Macaulay ring, but $K_{\mathcal{R}}$ is not a Cohen-Macaulay $\mathcal{R}$ -module. Therefore $K_{\mathcal{R}_{\mathfrak{n}}}$ is not a Cohen-Macaulay $\mathcal{R}_{\mathfrak{n}}$ -module where $\mathfrak{n}$ denotes the graded maximal ideal of $\mathcal{R}$, although the Noetherian local domain $\mathcal{R}_{\mathfrak{n}}$ possesses a birational Cohen-Macaulayfication.

It is well-known that M is Cohen-Macaulay if and only if $I(x_1, \dots, x_d; M) = 0$ for some (for all) s.o.p. $(x_1, \dots, x_d)$ of M , where

$$I(x_1, \dots, x_d; M) := \ell(M/(x_1, \dots, x_d)M) - e(x_1, \dots, x_d; M)$$

and $e(x_1, \dots, x_d; M)$ is the multiplicity of M with respect to $(x_1, \dots, x_d)$. Moreover, if $x \in \mathfrak{m}$ is an M -regular element, then M is Cohen-Macaulay if and only if so is M/xM .

It is clear that if M is Cohen-Macaulay, then so is K_M . The converse statement is not true, see Theorem 1.4(b). Note that K_M satisfies the condition Serre (S_2) . Therefore K_M is Cohen-Macaulay whenever $d \leq 2$. In case where $d \geq 3$, we have the following characterizations for the canonical module to be Cohen-Macaulay.

Lemma 2.6. ([Nh, Theorem 4.2], [BN, Theorem 2.5]). *The following statements are true.*

- (a) K_M is Cohen-Macaulay if and only if $\text{Rl}(H_{\mathfrak{m}}^{d-k-1}(M/(x_1, \dots, x_k)M)) = 0$ for a (and for all) strict $\mathfrak{f}$ -sequence $(x_1, \dots, x_d)$ of M and all $k = 0, \dots, d-3$.
- (b) If $d \geq 4$, then K_M is Cohen-Macaulay if and only if $K_{M/xM}$ is Cohen-Macaulay for every strict $\mathfrak{f}$ -element x of M .

Following Cuong, Schenzel and Trung [CST], M is said to be *generalized Cohen-Macaulay* if $I(M) := \sup I(x_1, \dots, x_d; M) < \infty$, where $(x_1, \dots, x_d)$ runs over all s.o.p. of M . Note that

M is generalized Cohen-Macaulay if and only if $\ell_R(H_{\mathfrak{m}}^i(M)) < \infty$ for all $i < d$. Note that K_M satisfies the condition Serre (S_2), therefore K_M is generalized Cohen-Macaulay whenever $d \leq 3$. In case where $d \geq 4$, we have the following characterizations for the canonical module to be generalized Cohen-Macaulay.

Lemma 2.7. ([LN, Main theorem]) *The following statements are equivalent:*

- (a) K_M is generalized Cohen-Macaulay.
- (b) There exists a number $c(M)$ such that $\text{Rl}(H_{\mathfrak{m}}^{d-k-1}(M/(x_1, \dots, x_k)M)) \leq c(M)$ for all strict $\mathfrak{f}$ -sequences $\underline{x} = (x_1, \dots, x_d)$ of M and all $k = 1, \dots, d-3$.
- (c) There exist a strict $\mathfrak{f}$ -sequence $\underline{x} = (x_1, \dots, x_d)$ of M and a number $c(\underline{x}, M)$ such that $\text{Rl}(H_{\mathfrak{m}}^{d-k-1}(M/(x_1^{n_1}, \dots, x_k^{n_k})M)) \leq c(\underline{x}, M)$ for all $k = 1, \dots, d-3$ and all positive integers $n_1, \dots, n_{d-3}$.

Furthermore, if the conditions (a), (b), (c) satisfy, then

$$\text{Rl}(H_{\mathfrak{m}}^{d-k-1}(M/(x_1, \dots, x_k)M)) \leq \sum_{i=0}^k \binom{k}{i} \ell(H_{\mathfrak{m}}^{i+2}(K_M))$$

for any $k = 1, \dots, d-3$. The equality holds true when $x_1, \dots, x_k \in \mathfrak{m}^{2^{k-1}q}$, where

$$q = \min\{t \in \mathbb{N} \mid \mathfrak{m}^t H_{\mathfrak{m}}^i(K_M) = 0 \text{ for all } i < d\}.$$

The notion of standard system of parameters (standard s.o.p. for short) defined in [Tr] (see also [Sch1]) is very important in the study of generalized Cohen-Macaulay modules. A s.o.p. $(x_1, \dots, x_d)$ of M is said to be a *standard s.o.p.* if

$$\ell_R(M/(x_1, \dots, x_d)M) - e(x_1, \dots, x_d; M) = \ell_R(M/(x_1^2, \dots, x_d^2)M) - e(x_1^2, \dots, x_d^2; M).$$

Then M is generalized Cohen-Macaulay if and only if there exists a standard s.o.p. of M . Note that if $(x_1, \dots, x_d)$ is a standard s.o.p. of M , then

$$I(x_1, \dots, x_d; M) = I(M) = \sum_{i=0}^{d-1} \binom{d-1}{i} \ell(H_{\mathfrak{m}}^i(M)).$$

In the introduction, we introduce the notion of canonical s.o.p. (see Definition 1.2), which will be used in the next section to characterize the generalized Cohen-Macaulayness of the canonical module. The following lemma gives a relation between standard s.o.p. of M and canonical s.o.p. of M .

Lemma 2.8. *If $(x_1, \dots, x_d)$ be a standard s.o.p. of M , then it is a canonical s.o.p of M .*

Proof. If $d \leq 2$, there is nothing to prove. Let $d \geq 3$. Suppose that $(x_1, \dots, x_d)$ is a standard s.o.p. of M . Then M is generalized Cohen-Macaulay, cf. [Tr]. Hence $\ell_R(K_M^i) < \infty$

for all $i < d$. So, each s.o.p. of M is an f-sequence of K_M^i for all i . It follows by Lemma 2.3(a) that each s.o.p. of M is a strict f-sequence. Since $(x_1, \dots, x_d)$ is a standard s.o.p. of M , so is $(x_1^2, \dots, x_d^2)$. Note that $M/(x_1, \dots, x_{d-3})M$ is generalized Cohen-Macaulay. Hence $\ell_R(H_{\mathfrak{m}}^2(M/(x_1, \dots, x_{d-3})M)) < \infty$. Similarly, $\ell_R(H_{\mathfrak{m}}^2(M/(x_1^2, \dots, x_{d-3}^2)M)) < \infty$. Therefore, we get by [Tr, Proposition 2.9] that

$$\begin{aligned} \text{Rl}(H_{\mathfrak{m}}^2(M/(x_1, \dots, x_{d-3})M)) &= \ell(H_{\mathfrak{m}}^2(M/(x_1, \dots, x_{d-3})M)) \\ &= \sum_{i=2}^{d-1} \binom{d-3}{i-1} \ell(H_{\mathfrak{m}}^i(M)) \\ &= \ell_R(H_{\mathfrak{m}}^2(M/(x_1^2, \dots, x_{d-3}^2)M)) \\ &= \text{Rl}(H_{\mathfrak{m}}^2(M/(x_1^2, \dots, x_{d-3}^2)M)). \end{aligned}$$

□

The converse statement of Lemma 2.8 is not true. In fact, by Theorem 1.4, there is an unmixed complete local ring R such that R is not generalized Cohen-Macaulay, but K_R is generalized Cohen-Macaulay. By Theorem 1.3, there is a canonical s.o.p. of R , but R does not admit a standard s.o.p.

3 Proof of Theorem 1.3

Before proving Theorem 1.3, we need some auxiliary lemmas.

For an Artinian R -module A , set $\dim_R A = \dim(R/\text{Ann}_R A)$. Note that A has a natural structure as an Artinian $\widehat{R}$ -module and $\dim_R A \geq \dim_{\widehat{R}} A$, see [CN, Proposition 2.4(ii), Corollary 4.7]. Moreover, $\ell_R(A) < \infty$ if and only if $\dim_R A = \dim_{\widehat{R}} A \leq 0$.

Since K_M satisfies the condition Serre (S_2) , we have $\dim_R H_{\mathfrak{m}}^i(K_M) \leq i - 2$ for all $i < d$ (see [Sch2, Propositions 2.2(c), 2.3(d)]). In particular, if $d \geq 3$, then $H_{\mathfrak{m}}^i(K_M) = 0$ for $i \leq 1$ and $\ell_R(H_{\mathfrak{m}}^2(K_M)) < \infty$.

Lemma 3.1. *Let $d \geq 4$, let $x \in \mathfrak{m}$ be a strict f-element of M . Then $\ell_R(0 :_{H_{\mathfrak{m}}^3(K_M)} x) < \infty$ and*

$$\text{Rl}(H_{\mathfrak{m}}^{d-2}(M/xM)) = \ell_R(H_{\mathfrak{m}}^2(K_M)/xH_{\mathfrak{m}}^2(K_M)) + \ell_R(0 :_{H_{\mathfrak{m}}^3(K_M)} x).$$

Proof. Set $N = M/xM$. Let y be a strict f-element of N . Then by Lemma 2.4(a) that

$$\text{Rl}(H_{\mathfrak{m}}^{d-2}(N)) = \ell_R(0 :_{K_N^{d-2}} y^n) < \infty$$

for all large enough integers n . Note that y^n is a strict f-element of N . Therefore, we have by Lemma 2.4(b) the exact sequence

$$0 \rightarrow K_N/y^n K_N \rightarrow K_N/y^n N \rightarrow (0 :_{K_N^{d-2}} y^n) \rightarrow 0.$$

Since $d \geq 4$ and $K_N/y^n N$ satisfies the condition Serre (S_2) , we have $\text{depth } K_N/y^n N \geq 2$. Since $\ell_R(0 :_{K_N^{d-2}} y^n) < \infty$, it follows by the above exact sequence that

$$(0 :_{K_N^{d-2}} y^n) = H_{\mathfrak{m}}^0(0 :_{K_N^{d-2}} y^n) \cong H_{\mathfrak{m}}^1(K_N/y^n K_N).$$

Since y^n is K_N -regular, we have the exact sequence

$$0 \rightarrow K_N \rightarrow K_N \rightarrow K_N/y^n K_N \rightarrow 0.$$

As $\dim K_N \geq 3$ and K_N satisfies the condition Serre (S_2) , we get $\text{depth } K_N \geq 2$. So,

$$H_{\mathfrak{m}}^1(K_N/y^n K_N) \cong (0 :_{H_{\mathfrak{m}}^2(K_N)} y^n).$$

Since $\ell_R(H_{\mathfrak{m}}^2(K_N)) < \infty$, it follows by Lemma 2.4(b) that

$$(0 :_{H_{\mathfrak{m}}^2(K_N)} y^n) = H_{\mathfrak{m}}^2(K_N) \cong H_{\mathfrak{m}}^2(K_M/xK_M)$$

for all large enough integers n . Therefore we get by all the aboves facts that

$$\text{Rl}(H_{\mathfrak{m}}^{d-2}(M/xM)) = \ell_R(H_{\mathfrak{m}}^2(K_M/xK_M)).$$

Hence $\ell_R(H_{\mathfrak{m}}^2(K_M/xK_M)) < \infty$. From the exact sequence

$$0 \rightarrow K_M \rightarrow K_M \rightarrow K_M/xK_M \rightarrow 0,$$

we have the exact sequence

$$0 \rightarrow H_{\mathfrak{m}}^2(K_M)/xH_{\mathfrak{m}}^2(K_M) \rightarrow H_{\mathfrak{m}}^2(K_M/xK_M) \rightarrow (0 :_{H_{\mathfrak{m}}^3(K_M)} x) \rightarrow 0.$$

Now, the result follows. $\square$

Lemma 3.2. *Let $d \geq 4$, let $(x_1, \dots, x_d)$ be an unconditioned strict f-sequence of M . Then*

$$\text{Rl}(H_{\mathfrak{m}}^{d-k-1}(M/(x_1^{n_1}, \dots, x_k^{n_k})M)) \leq \text{Rl}(H_{\mathfrak{m}}^{d-k-1}(M/(x_1^{m_1}, \dots, x_k^{m_k})M))$$

for all integers $1 \leq k \leq d-3$ and all positive integers $n_i \leq m_i$ for $i = 1, \dots, k$.

Proof. We prove the lemma by induction on d .

Let $d = 4$. Then $k = 1$. We have by Lemma 3.1 that

$$\begin{aligned} \text{Rl}(H_{\mathfrak{m}}^{d-2}(M/x^n M)) &= \ell_R(H_{\mathfrak{m}}^2(K_M)/x^n H_{\mathfrak{m}}^2(K_M)) + \ell_R(0 :_{H_{\mathfrak{m}}^3(K_M)} x^n) \\ &\leq \ell_R(H_{\mathfrak{m}}^2(K_M)/x^m H_{\mathfrak{m}}^2(K_M)) + \ell_R(0 :_{H_{\mathfrak{m}}^3(K_M)} x^m) \\ &= \text{Rl}(H_{\mathfrak{m}}^{d-2}(M/x^m M)). \end{aligned}$$

Assume that $d > 4$. Set $N = M/(x_2^{n_2}, \dots, x_k^{n_k})M$ and $L = M/x_1^{m_1}M$. Then $\dim L \geq 4$ and $\dim N = d - k + 1 \geq 4$. Since $(x_1, \dots, x_d)$ is an unconditioned strict f-sequence of M , it

follows by Lemma 2.3(b) that $(x_1, x_2^{n_2}, \dots, x_k^{n_k})$ is also an unconditioned strict f-sequence of M . Hence, x_1 is a strict f-element of N . Therefore we get

$$\begin{aligned} \text{Rl} \left(H_{\mathfrak{m}}^{d-k-1}(M/(x_1^{n_1}, \dots, x_k^{n_k})M) \right) &= \text{Rl} \left(H_{\mathfrak{m}}^{(d-k+1)-2}(N/x_1^{n_1}N) \right) \\ &\leq \text{Rl} \left(H_{\mathfrak{m}}^{(d-k+1)-2}(N/x_1^{m_1}N) \right) \\ &= \text{Rl} \left(H_{\mathfrak{m}}^{d-k-1}(M/(x_1^{m_1}, x_2^{n_2}, \dots, x_k^{n_k})M) \right) \\ &= \text{Rl} \left(H_{\mathfrak{m}}^{d-k-1}(L/(x_2^{n_2}, \dots, x_k^{n_k})L) \right). \end{aligned}$$

It is clear that $(x_2, \dots, x_k)$ is an unconditioned strict f-sequence of L and $\dim L = d - 1$. So, we get by induction hypothesis that

$$\begin{aligned} \text{Rl} \left(H_{\mathfrak{m}}^{d-k-1}(L/(x_2^{n_2}, \dots, x_k^{n_k})L) \right) &= \text{Rl} \left(H_{\mathfrak{m}}^{(d-1)-(k-1)-1}(L/(x_2^{n_2}, \dots, x_k^{n_k})L) \right) \\ &\leq \text{Rl} \left(H_{\mathfrak{m}}^{(d-1)-(k-1)-1}(L/(x_2^{m_2}, \dots, x_k^{m_k})L) \right) \\ &= \text{Rl} \left(H_{\mathfrak{m}}^{d-k-1}(M/(x_1^{m_1}, x_2^{m_2}, \dots, x_k^{m_k})M) \right). \end{aligned}$$

□

The following property of Artinian module is useful in the proof of Theorem 1.3. Let A be an Artinian R -module. It follows by [Ro, Theorem 6] and [CN, Corollary 4.7] that

$$\dim_{\widehat{R}} A = \inf \{ t \in \mathbb{N} \mid \exists x_1, \dots, x_t \in \mathfrak{m} \text{ such that } \ell_R(0 :_A (x_1, \dots, x_t)) < \infty \}.$$

A system $(x_1, \dots, x_t)$ of elements in $\mathfrak{m}$ (where $t = \dim_{\widehat{R}} A$) is said to be a *system of parameters* of A if $\ell_R(0 :_A (x_1, \dots, x_t)) < \infty$. It is clear that if $(x_1, \dots, x_t)$ is a system of parameters of A , then $\dim_{\widehat{R}}(0 :_A (x_1, \dots, x_n)) = t - n$ for all $n \leq t$. If $\dim_{\widehat{R}} A > 0$ and $x \in \mathfrak{m}$ be such that $\dim_{\widehat{R}}(0 :_A x) = \dim_{\widehat{R}} A - 1$, then x is said to be a *parameter* of A .

Lemma 3.3. *Let A be an Artinian R -module. If $\dim_{\widehat{R}} A > 0$ and x is a parameter of A , then for all positive integers n we have*

$$(0 :_A x^n) \neq (0 :_A x^{n+1}).$$

Proof. Assume in contrary that $(0 :_A x^n) = (0 :_A x^{n+1})$ for some integer $n > 0$. We claim that $A = (0 :_A x^n)$. In fact, let $a \in A$. Since A is $\mathfrak{m}$ -torsion, we have $\mathfrak{m}^s a = 0$ for some integer $s > 0$. Hence $x^s a = 0$. If $s \leq n$, then $a \in (0 :_A x^n)$. So, we assume that $s \geq n + 1$. Then we have $x^{n+1}(x^{s-n-1}a) = 0$. Hence $x^{s-n-1}a \in (0 :_A x^{n+1}) = (0 :_A x^n)$. Therefore $x^{s-1}a = 0$. Continue this process, after some steps we have $x^{n+1}a = 0$. Hence $a \in (0 :_A x^{n+1}) = (0 :_A x^n)$. Therefore, $A = (0 :_A x^n)$ and the claim is proved. Note that x^n is also a parameter of A . Since $\dim_{\widehat{R}} A > 0$, we have by the claim that

$$\dim_{\widehat{R}} A = \dim_{\widehat{R}}(0 :_A x^n) = \dim_{\widehat{R}} A - 1.$$

This gives a contradiction. □

Corollary 3.4. *Let $d \geq 4$ and let $x \in \mathfrak{m}$ be a strict f-element of M such that*

$$\text{Rl} \left(H_{\mathfrak{m}}^{d-2}(M/xM) \right) = \text{Rl} \left(H_{\mathfrak{m}}^{d-2}(M/x^2M) \right).$$

Then $\ell_R(H_m^3(K_M)) < \infty$, $xH_m^i(K_M) = 0$ for all $i \leq 3$, and

$$\text{Rl}(H_m^{d-2}(M/xM)) = \text{Rl}(H_m^{d-2}(M/x^n M)) = \ell_R(H_m^2(K_M)) + \ell_R(H_m^3(K_M))$$

for all $n > 0$. In particular, if $d = 4$, then M is generalized Cohen-Macaulay canonical.

Proof. Since K_M satisfies the condition Serre (S_2) , it follows that $\ell_R(H_m^2(K_M)) < \infty$ and $\dim_R H_m^3(K_M) \leq 1$. By Lemma 3.1 and by our assumption, $xH_m^2(K_M) = x^2H_m^2(K_M)$ and $(0 :_{H_m^3(K_M)} x) = (0 :_{H_m^3(K_M)} x^2)$. It follows by Nakayama Lemma that $xH_m^2(K_M) = 0$. Next, we claim that $\ell_R(H_m^3(K_M)) < \infty$. In fact, suppose $\ell_R(H_m^3(K_M)) = \infty$. Then $\dim_R H_m^3(K_M) = 1$. Hence $\dim_{\hat{R}} H_m^3(K_M) = 1$ by [CNN, Proposition 2.4, Corollary 4.2(iii)]. Since $\ell_R(0 :_{H_m^3(K_M)} x) < \infty$ by Lemma 3.1, it follows that x is a parameter of $H_m^3(K_M)$. Hence $(0 :_{H_m^3(K_M)} x) \neq (0 :_{H_m^3(K_M)} x^2)$ by Lemma 3.3. This gives a contradiction, and the claim is proved. Since $(0 :_{H_m^3(K_M)} x) = (0 :_{H_m^3(K_M)} x^2)$, we get by the same arguments as in the proof of Lemma 3.3 that $H_m^3(K_M) = (0 :_{H_m^3(K_M)} x)$. So, $xH_m^3(K_M) = 0$. Now, the rest statement follows by Lemma 3.1. $\square$

Lemma 3.5. Suppose that $d \geq 4$. Let $\underline{x} = (x_1, \dots, x_k)$ be a strict f-sequence of M , where $1 \leq k \leq d-3$ is an integer. Then, there exists a positive integer $m(\underline{x})$ such that

$$\text{Rl}(H_m^{d-k}(M/(x_1, \dots, x_{k-1})M)) \leq \text{Rl}(H_m^{d-k-1}(M/(x_1, \dots, x_{k-1}, x_k^{m(\underline{x})})M)).$$

Proof. Set $N := M/(x_1, \dots, x_{k-1})M$. We can choose a positive integer $m(\underline{x})$ such that $x_k^{m(\underline{x})}H_m^0(K_N^{d-k}) = 0$. Note that $x_k^{m(\underline{x})}$ is a filter regular element of K_N^{d-k} by Lemma 2.3(a), i.e. it is K_N^{d-k} -regular in dimension > 0 in sense of [BN1]. Therefore, we have by [DN, Lemma 2.3] the following exact sequence

$$0 \rightarrow H_m^0(K_N^{d-k}) \rightarrow H_m^0(K_N^{d-k}/x_k^{m(\underline{x})}K_N^{d-k}) \rightarrow (0 :_{H_m^1(K_N^{d-k})} x_k^{m(\underline{x})}) \rightarrow 0.$$

Since $x_k^{m(\underline{x})}$ is a strict f-element of N , we have by Lemma 2.4(b) the following exact sequence

$$0 \rightarrow K_N^{d-k}/x_k^{m(\underline{x})}K_N^{d-k} \rightarrow K_{N/x_k^{m(\underline{x})}N}^{d-k-1} \rightarrow (0 :_{K_N^{d-k-1}} x_k^{m(\underline{x})}) \rightarrow 0.$$

Therefore, it follows by Lemma 2.4(a) that

$$\begin{aligned} \text{Rl}(H_m^{d-k}(M/(x_1, \dots, x_{k-1})M)) &= \ell_R(H_m^0(K_N^{d-k})) \\ &\leq \ell_R(H_m^0(K_N^{d-k}/x_k^{m(\underline{x})}K_N^{d-k})) \\ &\leq \ell_R(H_m^0(K_{N/x_k^{m(\underline{x})}N}^{d-k-1})) \\ &= \text{Rl}(H_m^{d-k-1}(M/(x_1, \dots, x_{k-1}, x_k^{m(\underline{x})})M)). \end{aligned}$$

$\square$

Now, we are ready to prove the first main result of this paper.

Proof of Theorem 1.3. (a) $\Rightarrow$ (d). By our assumption (a), $\ell_R(H_{\mathfrak{m}}^i(K_M)) < \infty$ for all $i < d$. Set

$$q = \min\{t \in \mathbb{N} \mid \mathfrak{m}^t H_{\mathfrak{m}}^i(K_M) = 0 \text{ for all } i < d\}.$$

Then there exists by Lemma 2.3 an unconditioned f-sequence $(x_1, \dots, x_d)$ of M contained in $\mathfrak{m}^{2^{d-4}q}$. We have by Lemma 2.7 that

$$\begin{aligned} \text{Rl}(H_{\mathfrak{m}}^2(M/(x_1, \dots, x_{d-3})M)) &= \sum_{i=0}^{d-3} \binom{d-3}{i} \ell_R(H_{\mathfrak{m}}^{i+2}K_M) \\ &= \text{Rl}(H_{\mathfrak{m}}^2(M/(x_1^2, \dots, x_{d-3}^2)M)). \end{aligned}$$

Therefore, $(x_1, \dots, x_d)$ is an unconditioned canonical s.o.p. of M .

(d) $\Rightarrow$ (c). Suppose that $(x_1, \dots, x_d)$ is an unconditioned canonical s.o.p. of M . It is enough to prove the following equalities

$$\text{Rl}(H_{\mathfrak{m}}^2(M/(x_1, \dots, x_{d-3})M)) = \text{Rl}(H_{\mathfrak{m}}^2(M/(x_1^{n_1}, \dots, x_{d-3}^{n_{d-3}})M))$$

for all positive integers $n_1, \dots, n_{d-3}$. We prove this by induction on d . If $d \leq 3$, there is nothing to prove. The case where $d = 4$ follows by Corollary 3.4.

Let $d > 4$ and assume that the result is valid for $d-1$. Let $n_1, \dots, n_k$ be positive integers. Set $N = M/x_1 M$ and $N' = M/x_1^2 M$. Then $\dim N = d-1 = \dim N'$. Since $(x_1, \dots, x_d)$ is an unconditioned canonical s.o.p. of M , we have by Lemma 3.2 that

$$\begin{aligned} \text{Rl}(H_{\mathfrak{m}}^2(N/(x_2, \dots, x_{d-3})N)) &\leq \text{Rl}(H_{\mathfrak{m}}^2(N/(x_2^2, \dots, x_{d-3}^2)N)) \\ &\leq \text{Rl}(H_{\mathfrak{m}}^2(M/(x_1^2, \dots, x_{d-3}^2)M)) \\ &= \text{Rl}(H_{\mathfrak{m}}^2(N/(x_2, \dots, x_{d-3})N)). \end{aligned}$$

Hence $\text{Rl}(H_{\mathfrak{m}}^2(N/(x_2, \dots, x_{d-3})N)) = \text{Rl}(H_{\mathfrak{m}}^2(N/(x_2^2, \dots, x_{d-3}^2)N))$. It follows that $(x_2, \dots, x_d)$ is an unconditioned canonical s.o.p. of N . Therefore, we get by induction hypothesis that

$$\text{Rl}(H_{\mathfrak{m}}^2(N/(x_2, \dots, x_{d-3})N)) = \text{Rl}(H_{\mathfrak{m}}^2(N/(x_2^{n_2}, \dots, x_{d-3}^{n_{d-3}})N))$$

for all positive integers $n_2, \dots, n_{d-3}$. Similarly, since $(x_1, \dots, x_d)$ is an unconditioned canonical s.o.p. of M , we have by Lemma 3.2 that

$$\text{Rl}(H_{\mathfrak{m}}^2(N'/(x_2, \dots, x_{d-3})N')) = \text{Rl}(H_{\mathfrak{m}}^2(N'/(x_2^2, \dots, x_{d-3}^2)N')).$$

Hence $(x_2, \dots, x_d)$ is an unconditioned canonical s.o.p. of N' . Therefore, we get by induction that

$$\text{Rl}(H_{\mathfrak{m}}^2(N'/(x_2, \dots, x_{d-3})N')) = \text{Rl}(H_{\mathfrak{m}}^2(N'/(x_2^{n_2}, \dots, x_{d-3}^{n_{d-3}})N')),$$

for any positive integers $n_2, \dots, n_{d-3}$. As $(x_1, \dots, x_d)$ is an unconditioned canonical s.o.p. of M , it follows by Lemma 3.2 that

$$\text{Rl}(H_{\mathfrak{m}}^2(N/(x_2, \dots, x_{d-3})N)) = \text{Rl}(H_{\mathfrak{m}}^2(N'/(x_2, \dots, x_{d-3})N')).$$

For given positive integers $n_2, \dots, n_{d-3}$, we have $\text{Rl}(H_{\mathfrak{m}}^2(L/x_1L)) = \text{Rl}(H_{\mathfrak{m}}^2(L/x_1^2L))$ by the above equalities, where $L = M/(x_2^{n_2}, \dots, x_k^{n_k})M$. Since x_1 is a strict f-element of L , we have by Corollary 3.4 that

$$\text{Rl}(H_{\mathfrak{m}}^2(L/x_1L)) = \text{Rl}(H_{\mathfrak{m}}^2(L/x_1^{n_1}L)),$$

the equalities are proved.

(c) $\Rightarrow$ (a). Assume that there exists a strict f-sequence $\underline{x} = (x_1, \dots, x_d)$ of M such that

$$c_{\underline{x}, M} := \sup_{n_1, \dots, n_{d-3} \in \mathbb{N}} \text{Rl}(H_{\mathfrak{m}}^2(M/(x_1^{n_1}, \dots, x_{d-3}^{n_{d-3}})M)) < \infty.$$

Let $\underline{n} = (n_1, \dots, n_{d-3})$ be a tuple of $d-3$ positive integers. Note that $(x_1^{n_1}, \dots, x_{d-3}^{n_{d-3}})$ is a strict f-sequence of M by Lemma 2.3(b). Therefore, there exists by Lemma 3.5 a positive integer $m(\underline{x}, \underline{n})$ such that

$$\text{Rl}(H_{\mathfrak{m}}^3(M/(x_1^{n_1}, \dots, x_{d-4}^{n_{d-4}})M)) \leq \text{Rl}(H_{\mathfrak{m}}^2(M/(x_1^{n_1}, \dots, x_{d-4}^{n_{d-4}}, x_{d-3}^{n_{d-3}m(\underline{x}, \underline{n})})M)).$$

It follows by our assumption that

$$\begin{aligned} \sup_{n_1, \dots, n_{d-4}} \text{Rl}(H_{\mathfrak{m}}^3(M/(x_1^{n_1}, \dots, x_{d-4}^{n_{d-4}})M)) &\leq \sup_{n_1, \dots, n_{d-3}} \text{Rl}(H_{\mathfrak{m}}^2(M/(x_1^{n_1}, \dots, x_{d-4}^{n_{d-4}}, x_{d-3}^{n_{d-3}m(\underline{x}, \underline{n})})M)) \\ &\leq \sup_{m_1, \dots, m_{d-3} \in \mathbb{N}} \text{Rl}(H_{\mathfrak{m}}^2(M/(x_1^{m_1}, \dots, x_{d-3}^{m_{d-3}})M)) < \infty. \end{aligned}$$

By the same arguments, we get

$$\sup_{n_1, \dots, n_k} \text{Rl}(H_{\mathfrak{m}}^{d-k-1}(M/(x_1^{n_1}, \dots, x_k^{n_k})M)) < \infty$$

for all $k = 1, \dots, d-3$. Therefore, K_M is generalized Cohen-Macaulay by Lemma 2.7(c) $\Rightarrow$ (a).

(a) $\Rightarrow$ (b) follows by Lemma 2.7 (a) $\Rightarrow$ (b).

(b) $\Rightarrow$ (c) is trivial.

Finally, let $(x_1, \dots, x_d)$ is an unconditioned canonical s.o.p. of M . Let $n > 2^{d-4}q$ be an integer, where q is the number defined from the beginning. Then, we get by Lemma 2.3 and by the proof of (d) $\Rightarrow$ (c) that

$$\begin{aligned} \text{Rl}(H_{\mathfrak{m}}^2(M/(x_1, \dots, x_{d-3})M)) &= \text{Rl}(H_{\mathfrak{m}}^2(M/(x_1^n, \dots, x_{d-3}^n)M)) \\ &= \sum_{i=0}^{d-3} \binom{d-3}{i} \ell_R(H_{\mathfrak{m}}^{i+2}(K_M)). \end{aligned}$$

□

4 Proof of Theorem 1.4

In this section, we keep the assumption that $(R, \mathfrak{m})$ is a Noetherian local ring which is a quotient of a Gorenstein local ring, M is a finitely generated R -module with $\dim M = d$. Let

K_M be the canonical module of M . For each integer $i \geq 0$, let K_M^i be the i -th deficiency module of M . The non Cohen-Macaulay locus of M , denoted by $\text{nCM}(M)$, is defined by

$$\text{nCM}(M) = \{\mathfrak{p} \in \text{Spec}(R) \mid M_{\mathfrak{p}} \text{ is not Cohen-Macaulay}\}.$$

Because R is a quotient of a Gorenstein local ring, $\text{nCM}(M)$ is closed under Zariski topology, see [CNN, Corollary 4.2](iv). Therefore, we can define its dimension $\dim \text{nCM}(M)$. If we stipulate that $\dim \emptyset = -1$, then M is Cohen-Macaulay if and only if $\dim \text{nCM}(M) = -1$. Moreover, M is generalized Cohen-Macaulay if and only if $\dim \text{nCM}(M) \leq 0$. In general, we have the following result.

Lemma 4.1. ([C, Theorems 3.1, 3.3]). $\dim \text{nCM}(M) \leq \max_{i < d} \dim_R H_{\mathfrak{m}}^i(M)$. The equality holds true if M is equidimensional.

It is clear that $\dim \text{nCM}(M) \leq d-1$. Moreover, $\dim \text{nCM}(M) = d-1$ if and only if M has an embedded prime of dimension $d-1$. Following M. Nagata [Na], M is said to be *unmixed* if $\dim(\widehat{R}/\widehat{\mathfrak{P}}) = \dim_{\widehat{R}} \widehat{M}$ for all $\widehat{\mathfrak{P}} \in \text{Ass}_{\widehat{R}} \widehat{M}$. Since R is a quotient of a Gorenstein local ring, it follows by [Mat, Theorem 23.2] that M is unmixed if and only if $\dim(R/\mathfrak{p}) = d$ for all $\mathfrak{p} \in \text{Ass}_R M$. Note that if M is unmixed, then $\dim \text{nCM}(M) \leq d-2$. For each integer $k \geq 1$, it should be noticed that if M satisfies the condition Serre (S_k) , then $\dim_R H_{\mathfrak{m}}^i(M) \leq i-k$ for all $i < d$, see [Sch2, Proposition 2.2](c). Therefore, by Lemma 4.1, we have the following consequence.

Corollary 4.2. If M satisfies the condition Serre (S_k) , then $\dim \text{nCM}(M) \leq d-k-1$. In particular, $\dim \text{nCM}(K_M) \leq d-3$.

Next, we study the non Cohen-Macaulay locus under a flat extension.

Lemma 4.3. Let $(S, \mathfrak{n})$ be a Noetherian local ring and $\varphi : R \rightarrow S$ a flat local homomorphism such that $S/\mathfrak{m}S$ is Cohen-Macaulay of dimension t . If M is not Cohen-Macaulay, then

$$\max_{i < d+t} \dim_S H_{\mathfrak{n}}^i(M \otimes_R S) = \dim(S/\mathfrak{m}S) + \max_{i < d} \dim_R H_{\mathfrak{m}}^i(M).$$

In addition, if M and $M \otimes_R S$ are equidimensional, then

$$\dim \text{nCM}(M \otimes_R S) = \dim(S/\mathfrak{m}S) + \dim \text{nCM}(M).$$

Proof. Set $\dim(S/\mathfrak{m}S) = t$. We have $\dim(M \otimes_R S) = d+t$ by [Mat, Theorem 15.1]. Since $S/\mathfrak{m}S$ is Cohen-Macaulay, it follows by [BS1, Theorem 2.1] that

$$H_{\mathfrak{n}}^i(M \otimes_R S) \cong H_{\mathfrak{n}}^t(H_{\mathfrak{m}}^{i-t}(M) \otimes_R S)$$

for all $i \geq t$. Moreover, according to [Mat, Theorem 23.3], we have

$$\text{depth}(M \otimes_R S) = \text{depth } M + \text{depth}(S/\mathfrak{m}S) \geq t.$$

Hence $H_{\mathfrak{n}}^i(M \otimes_R S) = 0$ for all integers $i < t$. We set $\mathfrak{a}(M) = \mathfrak{a}_0(M) \cdot \mathfrak{a}_1(M) \cdots \mathfrak{a}_{d-1}(M)$ and $\mathfrak{a}(M \otimes_R S) = \mathfrak{a}_0(M \otimes_R S) \cdot \mathfrak{a}_1(M \otimes_R S) \cdots \mathfrak{a}_{d+t-1}(M \otimes_R S)$, where $\mathfrak{a}_i(M) = \text{Ann}_R H_{\mathfrak{m}}^i(M)$

and $\mathfrak{a}_i(M \otimes_R S) = \text{Ann}_S H_{\mathfrak{n}}^i(M \otimes_R S)$ for all i . Then $\mathfrak{a}(M)S \subseteq \mathfrak{a}(M \otimes_R S)$ by the above isomorphism. Since M is not Cohen-Macaulay, it follows by [C, Theorem 3.1(i)] and by the same arguments as in the proof of [C, Theorem 5.1] that

$$\max_{i < d+t} \dim_S H_{\mathfrak{n}}^i(M \otimes_R S) = \dim(S/\mathfrak{m}S) + \max_{i < d} \dim_R H_{\mathfrak{m}}^i(M).$$

The rest statement follows by this equality and by Lemma 4.1. $\square$

Let $t > 0$ be an integer, let $S = R[[x_1, \dots, x_t]]$ be the formal power series ring of t variables over R . Then the natural map $R \rightarrow S$ is flat local and the fiber ring $S/\mathfrak{m}S$ is Cohen-Macaulay. So, R is Cohen-Macaulay if and only if so is S . The following lemma shows some relations between the canonical modules and the deficiency modules of R and that of S . The proof of this lemma given below was suggested by P. Schenzel.

Lemma 4.4. *Let $S = R[[x_1, \dots, x_t]]$ be the formal power series ring over R . Then*

- (a) *K_R is Cohen-Macaulay if and only if so is K_S . If K_R is not Cohen-Macaulay, then $\dim \text{nCM}(K_S) = t + \dim \text{nCM}(K_R)$.*
- (b) *$K_S^i \cong K_R^{i-t} \otimes_R S$ for all $i \geq t$ and $K_S^i = 0$ for all $i < t$. In particular, if $K_R^{i-t} \neq 0$, then $\dim_S K_S^i = t + \dim_R K_R^{i-t}$.*

Proof. (a) Since the ring $S/\mathfrak{m}S$ is Gorenstein, $K_S \cong K_R \otimes_R S$ by [AG, Theorem 4.1]. It is clear that the natural injection $R \rightarrow S$ is a local flat homomorphism. Since $\dim S/\mathfrak{m}S = t$, we have $\dim K_S = t + \dim K_R$. Because $\text{depth}(S/\mathfrak{m}S) = t$, it follows that $\text{depth } K_S = t + \text{depth } K_R$. Therefore, K_R is Cohen-Macaulay if and only if so is K_S . Suppose that K_R is not Cohen-Macaulay. Note that K_R and K_S are equidimensional. So, we get by Lemma 4.3 that

$$\dim \text{nCM}(K_S) = \dim \text{nCM}(K_R \otimes_R S) = t + \dim \text{nCM}(K_R).$$

(b) Let $(R', \mathfrak{m}')$ be a Gorenstein local ring such that R is a factor ring of R' . Set $R = R'/J$ for some ideal J of R' . Suppose that $\dim R' = n$. The Local Duality Theorem (see [BS, 11.2.6]) provides the natural isomorphisms

$$H_{\mathfrak{m}}^i(R) \cong \text{Hom}_R(K_R^i, E(R/\mathfrak{m})) = \text{Hom}_R(\text{Ext}_{R'}^{n-i}(R, R'), E(R/\mathfrak{m}))$$

for all $i \in \mathbb{N}$. Let $S' = R'[[x_1, \dots, x_t]]$ be the formal power series ring of t variables over R' . For each integer $i \geq 0$, we have the isomorphism

$$\text{Ext}_{R'}^i(R, R') \otimes_{R'} S' \cong \text{Ext}_{S'}^i(R \otimes_{R'} S', S').$$

Note that S' is a Gorenstein ring with $\dim S' = n+t$. This implies the following isomorphisms

$$K_R^i \otimes_{R'} S' \cong K_{R \otimes_{R'} S'}^{i+t}$$

for all integers $i \geq 0$. Now K_R^i and R have the structure of an R' -module. So,

$$R \otimes_{R'} S' \cong R \otimes_R R'/J \otimes_{R'} S' \cong R \otimes_R S \cong S;$$

$$K_R^i \otimes_{R'} S' \cong K_R^i \otimes_R R'/J \otimes_{R'} S' \cong K_R^i \otimes_R S.$$

Thus, $K_S^i \cong K_R^{i-t} \otimes_R S$ for all $i \geq t$. It is clear that $K_S^i = 0$ for all $i < t$. Therefore, if $K_R^{i-t} \neq 0$, then $\dim_S K_S^i = t + \dim_R K_R^{i-t}$. $\square$

In order to prove Theorem 1.4, we need recall the notion of idealization introduced by M. Nagata [Na]. We can make Cartesian product $R \times M$ into a ring with respect to the componentwise addition and the multiplication defined by

$$(r_1, m_1)(r_2, m_2) = (r_1 r_2, r_1 m_2 + r_2 m_1).$$

This ring is called the *idealization* of M over R , and denoted by $R \ltimes M$. Note that $R \ltimes M$ is a commutative Noetherian local ring with the identity $(1, 0)$. The unique maximal ideal of $R \ltimes M$ is $\mathfrak{m} \ltimes M$. Note that $K_R, K_M, K_{R \ltimes M}$ are equidimensional. Therefore the following lemma can be verified by Lemma 4.1, [C, Theorem 3.1] and [L].

Lemma 4.5. *The following statements are true*

- (a) *If $\dim M < \dim R$, then $\dim \text{nCM}(K_{R \ltimes M}) = \dim \text{nCM}(K_R)$.*
- (b) *If $\dim M = \dim R$, then $\dim \text{nCM}(K_{R \ltimes M}) = \max\{\dim \text{nCM}(K_R), \dim \text{nCM}(K_M)\}$.*

Proof of Theorem 1.4. (a) Note that $\dim \text{nCM}(K_M) \leq d - 3$ by Corollary 4.2. So, it is enough to prove that $\dim \text{nCM}(K_M) \leq \dim \text{nCM}(M)$. If K_M is Cohen-Macaulay, then there is nothing to prove. Assume that K_M is not Cohen-Macaulay. Set $s = \dim \text{nCM}(K_M)$. Then there exists $\mathfrak{p} \in \text{nCM}(K_M)$ such that $\dim R/\mathfrak{p} = s$. Hence $(K_M)_{\mathfrak{p}}$ is not Cohen-Macaulay. It follows that $\mathfrak{p} \in \text{Supp}(K_M)$. Note that $\text{Ass}(K_M) = \{\mathfrak{p} \in \text{Ass}(M) \mid \dim R/\mathfrak{p} = d\}$. Therefore $\dim R/\mathfrak{p} + \dim M_{\mathfrak{p}} = d$. Hence $(K_M)_{\mathfrak{p}} \cong K_{M_{\mathfrak{p}}}$, see [Sch2, Proposition 2.2](b). So $K_{M_{\mathfrak{p}}}$ is not Cohen-Macaulay. It follows that $M_{\mathfrak{p}}$ is not Cohen-Macaulay. Hence $\dim \text{nCM}(M) \geq s$.

(b) Let $d \geq 3$ be an integer. Let r, s be integers such that $-1 \leq s \leq d-3$ and $s \leq r \leq d-2$. We consider the following two cases.

- The case where $s = -1$. If $r = -1$, then any Cohen-Macaulay complete local ring of dimension d satisfies the requirement. Assume that $r \geq 0$. Let $(R_1, \mathfrak{m}_1)$ be a Buchsbaum complete local ring such that $\dim R_1 = d - r \geq 2$, $H_{\mathfrak{m}_1}^1(R_1) \neq 0$ and $H_{\mathfrak{m}_1}^i(R_1) = 0$ for $i \neq d$ and $i \neq 1$ (such a local ring R_1 exists by the construction of S. Goto [Go]). Then R_1 is not Cohen-Macaulay. Hence $\dim \text{nCM}(R_1) = 0$. Note that R_1 is generalized Cohen-Macaulay. Therefore, it follows by [BN, Corollary 2.7] that K_{R_1} is Cohen-Macaulay. Hence $\dim \text{nCM}(K_{R_1}) = -1$. Let $R = R_1[[x_1, \dots, x_r]]$ be the formal power series ring of r variables over R_1 . Then, R is a Noetherian complete local ring and $\dim R = d$. Because K_{R_1} is Cohen-Macaulay, it follows by Lemma 4.4 that K_R is Cohen-Macaulay, i.e. $\dim \text{nCM}(K_R) = -1 = s$. Since R_1 is Buchsbaum and $H_{\mathfrak{m}_1}^0(R_1) = 0$, it follows that R_1 is unmixed, i.e. $\dim(R_1/\mathfrak{p}) = \dim R_1$ for all $\mathfrak{p} \in \text{Ass} R_1$. Since R_1 is not Cohen-Macaulay, we have by Lemma 4.4 that $\dim \text{nCM}(R) = r + \dim \text{nCM}(R_1) = r$. For each $\mathfrak{p} \in \text{Spec}(R_1)$, since $R/\mathfrak{p}R \cong (R_1/\mathfrak{p})[[x_1, \dots, x_r]]$ is a domain, it follows that $\mathfrak{p}R \in \text{Spec}(R)$ and $\dim(R/\mathfrak{p}R) = r + \dim(R_1/\mathfrak{p})$. Therefore, we have by the flatness of the natural injection $R_1 \rightarrow R$ and by

[Mat, Theorem 23.2] that

$$\text{Ass } R = \bigcup_{\mathfrak{p} \in \text{Ass } R_1} \text{Ass}_R(R/\mathfrak{p}R) = \{\mathfrak{p}R \mid \mathfrak{p} \in \text{Ass } R_1\}.$$

It follows that for each $\mathfrak{P} \in \text{Ass } R$, there exists $\mathfrak{p} \in \text{Ass } R_1$ such that $\mathfrak{P} = \mathfrak{p}R$. Hence $\dim(R/\mathfrak{P}) = r + \dim(R_1/\mathfrak{p}) = r + \dim R_1 = \dim R = d$. Therefore, R is a unmixed complete local ring which satisfies the requirement.

• The left case where $0 \leq s \leq r$. Let $(R_2, \mathfrak{m}_2)$ be a Buchsbaum complete local ring such that $\dim R_2 = d - s \geq 3$, $H_{\mathfrak{m}_2}^0(R_2) = 0$ and $H_{\mathfrak{m}_2}^{d-s-1}(R_2) \neq 0$ (such a local ring R_2 exists by the construction of S. Goto [Go]). It is clear that $\dim \text{nCM}(R_2) = 0$. Moreover, K_{R_2} is generalized Cohen-Macaulay. Note that $\text{Rl}(H_{\mathfrak{m}_2}^{d-s-1}(R_2)) = \ell_{R_2}(H_{\mathfrak{m}_2}^{d-s-1}(R_2)) \neq 0$. Therefore, it follows by Lemma 2.6(a) that K_{R_2} is not Cohen-Macaulay. Hence $\dim \text{nCM}(K_{R_2}) = 0$. Let $R_3 = R_2[[x_1, \dots, x_s]]$ be the formal power series ring of s variables over R_2 . By the same arguments in the above, we can show that R_3 is a Noetherian unmixed complete local ring with the unique maximal ideal $\mathfrak{n} = (\mathfrak{m}_2, x_1, \dots, x_s)R_3$ and $\dim R_3 = d$. Since R_2 and K_{R_2} are not Cohen-Macaulay, we get by Lemma 4.4 that

$$\begin{aligned} \dim \text{nCM}(K_{R_3}) &= s + \dim \text{nCM}(K_{R_2}) = s; \\ \dim \text{nCM}(R_3) &= s + \dim \text{nCM}(R_2) = s. \end{aligned}$$

Therefore, if $s = r$, then we set $R = R_3$ and the ring R satisfies the requirement. Now, we can assume that $s < r$. It is clear that $H_{\mathfrak{n}}^i(R_3) = 0$ for all $i < s$. For each integer $i \geq s$, we have $K_{R_3}^i \cong K_{R_2}^{i-s} \otimes_{R_2} R_3$ by Lemma 4.4(b). Note that the natural map $R_2 \rightarrow R_3$ is flat. Therefore, for any integer $i < d$, we get by Local Duality Theorem (see [BS, 11.2.6]) that if $H_{\mathfrak{n}}^i(R_3) \neq 0$, then $H_{\mathfrak{m}_2}^{i-s}(R_2) \neq 0$ and

$$\dim_{R_3} H_{\mathfrak{n}}^i(R_3) = \dim_{R_2} H_{\mathfrak{m}_2}^{i-s}(R_2) + s = s.$$

Therefore, $\dim_{R_3} H_{\mathfrak{n}}^i(R_3) \leq s$, for all $i < d$. Let $a_1, \dots, a_{d-r}$ be a part of a system of parameters of R_3 . Set $P = (a_1, \dots, a_{d-r})R_3$ and $Q = R_3/(a_1, \dots, a_{d-r})R_3$. Then we have the following exact sequence of R_3 -modules

$$0 \rightarrow P \rightarrow R_3 \rightarrow Q \rightarrow 0.$$

So, we have the following sequences

$$H_{\mathfrak{n}}^i(R_3) \rightarrow H_{\mathfrak{n}}^i(Q) \rightarrow H_{\mathfrak{n}}^{i+1}(P) \rightarrow H_{\mathfrak{n}}^{i+1}(R_3),$$

for all i . Note that $\dim Q = r \leq d - 2$ and $\dim P = \dim R_3 = d$. Hence $\dim_{R_3} H_{\mathfrak{n}}^r(Q) = r$ and $\dim_{R_3} H_{\mathfrak{n}}^i(Q) < r$ for all $i \neq r$. Since $r \leq d - 2$ and $\dim_{R_3} H_{\mathfrak{n}}^i(R_3) \leq s$ for all $i < d$, we have $\dim_{R_3} H_{\mathfrak{n}}^r(R_3) \leq s < r$ and $\dim_{R_3} H_{\mathfrak{n}}^{r+1}(R_3) \leq s < r$. Hence $\dim_{R_3} H_{\mathfrak{n}}^{r+1}(P) = r$ and $\dim_{R_3} H_{\mathfrak{n}}^i(P) < r$ for all $i \neq r + 1$ and $i \neq d$. Since R_3 is unmixed, P is unmixed. Therefore, we get by Lemma 4.1 that

$$\dim \text{nCM}(P) = \max_{i < d} \dim_{R_3} H_{\mathfrak{n}}^i(P) = r.$$

Let $R = R_3 \ltimes P$ be the idealization of the R_3 -module P . Then R is a Noetherian local ring with the unique maximal ideal $\mathfrak{m} = \mathfrak{n} \times P$. Since R_3 is complete under $\mathfrak{n}$ -adic topology, R is complete under $\mathfrak{m}$ -adic topology, see [AW, Theorem 4.11]. Since R_3 and P are unmixed of dimension d , we can check that R is unmixed of dimension d . Since R is completed and unmixed, we have by Lemma 4.1 that

$$\dim \text{nCM}(R) = \max_{i < d} \dim_R H_{\mathfrak{m}}^i(R).$$

Consider the following exact sequence $0 \rightarrow P \xrightarrow{\epsilon} R \xrightarrow{\rho} R_3 \rightarrow 0$, where $\epsilon(x) = (0, x)$ for all $x \in P$ and $\rho(a, x) = a$ for all $(a, x) \in R$. From the induced long exact sequence of local cohomology modules, we can check that

$$\dim_R H_{\mathfrak{m}}^{r+1}(R) = \dim_R H_{\mathfrak{m}}^{r+1}(P) = \dim_{R_3} H_{\mathfrak{n}}^{r+1}(P) = r$$

and $\dim_R H_{\mathfrak{m}}^i(R) \leq s < r$ for all $i \neq r+1$ and $i \neq d$. Thus, $\dim \text{nCM}(R) = r$. Since $r \leq d-2$ and $\dim_{R_3} Q = r$, we have by the exact sequence $0 \rightarrow P \rightarrow R_3 \rightarrow Q \rightarrow 0$ that $K_P \cong K_{R_3}$. Moreover, since K_{R_2} is not Cohen-Macaulay, we have by Lemma 4.4(a) that $\dim \text{nCM}(K_{R_3}) = s + \dim \text{nCM}(K_{R_2}) = s$. Hence, by Lemma 4.1 and Lemma 4.5, we have

$$\dim \text{nCM}(K_R) = \max\{\dim \text{nCM}(K_{R_3}), \dim \text{nCM}(K_P)\} = \dim \text{nCM}(K_{R_3}) = s.$$

□

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References

- [AW] D. D. Anderson, M. Winders, *Idealization of a module*, J. Commutative Algebra, **1** (2009), 1-55.
- [A] Y. Aoyama, *On the depth and the projective dimension of the canonical module*, Japan. J. Math., **6** (1980), 61-66.
- [AG] Y. Aoyama, S. Goto, *On the endomorphism ring of the canonical module*, J. Math. Kyoto Univ., **25** (1985), 21-30.
- [BS] M. Brodmann, R. Y. Sharp, “Local cohomology: an algebraic introduction with geometric applications”, Cambridge University Press, 1998.
- [BS1] M. Brodmann, R. Y. Sharp, *On the dimension and multiplicity of local cohomology modules*, Nagoya Math. J., **167** (2002), 217-233.
- [BN] M. Brodmann, L. T. Nhan, *On canonical Cohen-Macaulay modules*, J. Algebra, **371** (2012), 480-491.
- [BN1] M. Brodmann, L. T. Nhan, *A finiteness result for associated primes of certain Ext-modules*, Comm. Algebra, **36** (2008), 1527-1536.

- [C] N. T. Cuong, *On the least degree of polynomials bounding above the differences between lengths and multiplicities of certain systems of parameters in local rings*, Nagoya Math. J., **125** (1992), 105-114.
- [CMN] N. T. Cuong, M. Morales, L. T. Nhan, *The finiteness of certain sets of attached prime ideals and the length of generalized fractions*, J. Pure Appl. Algebra, **189**, (2004), 109-121.
- [CN] N. T. Cuong, L. T. Nhan, *On the Noetherian dimension of Artinian modules*, Vietnam J. Math., **30** (2002), 121-130.
- [CNN] N. T. Cuong, L. T. Nhan, N. T. K. Nga, *On pseudo supports and non Cohen-Macaulay locus of a finitely generated module*, J. Algebra, **323** (2010), 3029-3038.
- [CST] N. T. Cuong, P. Schenzel, N. V. Trung, *Verallgemeinerte Cohen - Macaulay moduln*, Math. Nachr., **85** (1978), 57-75.
- [DN] T. D. Dung, L. T. Nhan, *A uniform bound of irreducibility index of good parameter ideals for certain class of modules*, Preprint.
- [L] N. T. H. Loan, *On canonical modules of idealizations*, Journal of Commutative Algebra, **9** (2017), 107-117.
- [LN] N. T. H. Loan, L. T. Nhan, *On generalized Cohen-Macaulay canonical modules*, Comm. Algebra, **41** (2013), 4453-4462.
- [Go] S. Goto, *On Buchsbaum ring*, J. Algebra, **67** (1980), 272-279.
- [GN] S. Goto, L. T. Nhan, *On the sequentially polynomial type of modules*, J. Math. Soc. Japan, **70** (2018), 363-383.
- [Mac] I. G. Macdonald, *Secondary representation of modules over a commutative ring*, Symposia Mathematica., **11** (1973).
- [Mat] H. Matsumura, "Commutative ring theory", Cambridge University Press, 1986.
- [Na] M. Nagata, "Local rings", Interscience, New York, 1962.
- [Nh] L. T. Nhan, *A remark on the monomial conjecture and Cohen-Macaulay canonical modules*, Proc. Amer. Math. Soc., **134** (2006), 2785-2794.
- [Ro] R. N. Roberts, *Krull dimension for Artinian modules over quasi local commutative rings*, Quart. J. Math. Oxford, **26** (1975), 269-273.
- [Sch1] P. Schenzel, *Standard systems of parameters and their blowing-up rings*, Journal für die reine und angewandte Mathematik, **344** (1983) 201-220.
- [Sch2] P. Schenzel, *On Birational Macaulayfications and Cohen-Macaulay canonical modules*, J. Algebra, **275** (2004), 751-770.
- [S] R. Y. Sharp, *Some results on the vanishing of local cohomology modules*, Proc. London Math. Soc., **30** (1975), 177-195.

- [SH] R. Y. Sharp, M. A. Hamieh, *Lengths of certain generalized fractions*, J. Pure Appl. Algebra, **38** (1985), 323-336.
- [Tr] N. V. Trung, *Toward a theory of generalized Cohen-Macaulay modules*, Nagoya Math. J., **102** (1986), 1-49.