Weak solutions to the complex Monge-Ampère equation on open subsets of $\mathbb{C}^n$ and applications

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Abstract

In the paper, we prove the existence of weak solutions to the complex Monge-Ampère equation in the class $\mathcal{D}(\Omega)$ on an open subset Ω of $\mathbb{C}^n$. As an application, we show the existence of a global solution of the complex Monge-Ampère equation $(dd^c u)^n = \mu$ in the class $\mathcal{D}(\mathbb{C}^n) \cap \mathcal{L}$ where μ is a Borel measure in $\mathbb{C}^n$.

1 Introduction

As well known, the complex Monge-Ampère operator plays a central role in pluripotential theory and has been extensively studied for many years. For a $\mathcal{C}^2$-smooth plurisubharmonic function u defined on an open subset Ω of $\mathbb{C}^n$, its complex Monge-Ampère operator is defined by

$$(dd^c u)^n = n!4^n det \left(\frac{\partial^2 u}{\partial z_j \partial \overline{z}_k} \right) dV_{2n},$$

where dV_{2n} is the volume form of $\mathbb{C}^n$. However, by an example in [19], Shiffman and Taylor have shown that it is impossible to extend the domain of definition of complex Monge-Ampère operator in a meaningful way to the whole class of plurisubharmonic functions and still have the range contained in the class of non-negative Borel measures (see Appendix 1 in [19]). Bedford and Taylor [3] proved in 1982 that the complex Monge-Ampère operator is defined for locally bounded plurisubharmonic functions. Cegrell in [9] introduced and investigated in 2004 some classes of unbounded plurisubharmonic functions on bounded hyperconvex domains in $\mathbb{C}^n$. He has shown that the complex Monge-Ampère operator is well defined on the class $\mathcal{E}(\Omega)$ as a non-negative Radon measure and it is continuous on decreasing sequences of plurisubharmonic functions in this class. At the same

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time, in [9] he proved that the class $\mathcal{E}(\Omega)$ is the biggest class of plurisubharmonic functions with this property. To extend the class $\mathcal{E}(\Omega)$, in 2006 Błocki in [6] introduced the class $\mathcal{D}(\Omega)$ of plurisubharmonic functions on an open Ω of $\mathbb{C}^n$ and showed that the complex Monge-Ampère operator can be well defined on this class. In the case $n = 2$ he described this class by the equality: $\mathcal{D}(\Omega) = PSH(\Omega) \cap W^{1,2}_{loc}(\Omega)$ (see Theorem 1.1 in [5]), where $W^{1,2}_{loc}(\Omega)$ is the Sobolev space on Ω. He also showed that if Ω is a bounded hyperconvex domain then $\mathcal{E}(\Omega) = PSH^-(\Omega) \cap \mathcal{D}(\Omega)$. Therefore, by results of Błocki in [6] one has known that the class $\mathcal{D}(\Omega)$ is the biggest class on which the complex Monge-Ampère operator is well-defined when Ω is an open subset of $\mathbb{C}^n$.
One of the important and central problems of pluripotential theory is to define weak solutions to the complex Monge-Ampère equation. Namely we consider the following problem. Let μ be a non-negative Radon measure defined in an open subset Ω of $\mathbb{C}^n$. To find $w \in PSH(\Omega)$ such that

$$\begin{cases} w \in \mathcal{D}(\Omega), \\ (dd^c w)^n = \mu \text{ in } \Omega. \end{cases} \tag{1}$$

When Ω is a strictly pseudoconvex domain in $\mathbb{C}^n$, there are some known results concerning to weak solutions of (1). Bedford and Taylor in [2] proved in 1976 that if $\mu = fdV$ with $f \in \mathcal{C}(\overline{\Omega})$ then (1) has a continuous solution on $\overline{\Omega}$. Kołodziej [18] showed in 1995 that if there exists a bounded subsolution for (1) then the problem is solvable.
In the case Ω is a bounded hyperconvex domain in $\mathbb{C}^n$, the problem becomes much more complicated. Cegrell [8] considered in 1998 weak solutions of the above problem for unbounded plurisubharmonic functions. In [9] he proved that if Ω is a bounded hyperconvex domain and μ vanishes on all pluripolar sets of Ω then (1) has a unique weak solution in the class $\mathcal{F}^a(\Omega)$ (see Theorem 5.14 in [9]). Next, Åhag, Cegrell, Czyż and Hiep in [1] studied in 2009 the above problem for non-negative measures carried by a pluripolar set. They showed that if $\mu \le (dd^c u)^n$ where $u \in \mathcal{E}(\Omega)$ then for every $H \in \mathcal{E}(\Omega) \cap MPSH(\Omega)$ there exists a weak solution $w \in \mathcal{E}(\Omega, H)$ such that $H + u \le w \le H$ satisfying (1), where $MPSH(\Omega)$ denotes the set of maximal plurisubharmonic functions on Ω.

The aim of this paper is to prove the existence of weak solutions to the equation (1) on an open subset of $\mathbb{C}^n$. Namely, we prove the following.

Theorem 1.1. *Let Ω be an open subset in $\mathbb{C}^n$ and $u \in \mathcal{D}(\Omega)$, $v \in MPSH(\Omega)$ be such that $u \le v$ in Ω. If $\mu \le (dd^c u)^n$ then there exists a solution w to (1) satisfying $u \le w \le v$ in Ω.*

As an application of Theorem 1.1, in Section 4 of this paper we study the existence of global solutions of the complex Monge-Ampère equation in $\mathbb{C}^n$. One of remarkable results of this section is the following.
Theorem 4.3. *Let μ be a non-negative Radon measure on $\mathbb{C}^n$ such that $\mu \le (dd^c log(1 + |z|^2))^n$ on $\mathbb{C}^n$. Assume that there exists $r \ge 2$ such that $\mu(|z| = r) = (dd^c log(1 + |z|^2))^n(|z| = r)$. Then there exists $w \in PSH(\mathbb{C}^n) \cap \mathcal{D}(\mathbb{C}^n)$ such that $(dd^c w)^n = \mu$.*
The organization of the paper is as follows. In Section 2 we recall some notions of pluripotential theory which is necessary for the next results of the paper. In

Section 3 we prove Theorem 1.1. Section 4 is devoted to the proof of the existence of global solutions of the complex Monge-Ampère equation in $\mathbb{C}^n$.

2 Preliminairies

In this section, we recall some elements of pluripotential theory that will be used throughout the paper. All this can be found in [1]-[18]. Let n be a positive integer and let Ω be an open set in $\mathbb{C}^n$. We denote by $PSH(\Omega)$ the family of plurisubharmonic functions defined on Ω and $PSH^-(\Omega)$ denotes the set of negative plurisubharmonic functions on Ω. We first recall the definition of the class $\mathcal{D}(\Omega)$ on an open set Ω in $\mathbb{C}^n$ (see [6]).

Definition 2.1. A plurisubharmonic function u defined on Ω belongs to $\mathcal{D}(\Omega)$ if there exists a nonnegative Radon measure μ on Ω such that if $\Omega' \Subset \Omega$ is an open subset and $\{u_j\} \subset PSH(\Omega') \cap \mathcal{C}^\infty(\Omega')$ is a sequence which decreases to u in Ω' then $(dd^c u_j)^n$ tends weakly to μ in Ω'. The measure μ we then denote by $(dd^c u)^n$.

Note that by results of Bedford - Taylor in [3] we have $PSH(\Omega) \cap L^\infty_{loc}(\Omega) \subset \mathcal{D}(\Omega)$. Moreover, if $n = 1$ then $SH(\Omega) = \mathcal{D}(\Omega)$. Next, we recall the following classes of plurisubharmonic functions introduced and investigated by Cegrell in [9] for the case Ω is a bounded hyperconvex domain in $\mathbb{C}^n$.

Definition 2.2. Let Ω be a bounded hyperconvex domain in $\mathbb{C}^n$. We say that a bounded, negative plurisubharmonic function φ on Ω belongs to $\mathcal{E}_0(\Omega)$ if $\{\varphi < -\varepsilon\} \Subset \Omega$ for all $\varepsilon > 0$ and $\int_\Omega (dd^c\varphi)^n < +\infty$.

Let $\mathcal{F}(\Omega)$ be the family of plurisubharmonic functions φ defined on Ω, such that there exists a decreasing sequence $\{\varphi_j\} \subset \mathcal{E}_0(\Omega)$ that converges pointwise to φ on Ω as $j \to \infty$ and

$$\sup_j \int_\Omega (dd^c\varphi_j)^n < \infty.$$

We denote by $\mathcal{E}(\Omega)$ the family of plurisubharmonic functions φ defined on Ω such that for every open set $G \Subset \Omega$ there exists a plurisubharmonic function $\psi \in \mathcal{F}(\Omega)$ satisfying $\psi \geq \varphi$ on Ω and $\psi = \varphi$ in G.

Definition 2.3. Let $\mathcal{K} \in \{\mathcal{F}, \mathcal{E}, \mathcal{D}\}$. We denote by $\mathcal{K}^a(\Omega)$ the subclass of $\mathcal{K}(\Omega)$ such that the complex Monge-Ampère measure $(dd^c.)^n$ vanishes on all pluripolar sets of Ω.

Let $f \in \mathcal{E}(\Omega)$ and $\mathcal{K} \in \{\mathcal{F}^a, \mathcal{E}^a, \mathcal{D}^a, \mathcal{F}, \mathcal{E}, \mathcal{D}\}$. Then we say that a plurisubharmonic function φ defined on Ω belongs to $\mathcal{K}(\Omega, f)$ if there exists a function $\psi \in \mathcal{K}(\Omega)$ such that

$$\psi + f \leq \varphi \leq f \text{ in } \Omega.$$

Proposition 2.4. *If Ω is a bounded hyperconvex domain in $\mathbb{C}^n$ then $\mathcal{E}(\Omega) = PSH^-(\Omega) \cap \mathcal{D}(\Omega)$.*

Proof. See Theorem 2.4 in [6]. □

Remark 2.5. (i) If Ω is an open subset of $\mathbb{C}^n$ and $u \in \mathcal{D}(\Omega)$ then $u|_{\mathbb{B}} - \sup_{\mathbb{B}} u \in \mathcal{E}(\mathbb{B})$ for all open ball $\mathbb{B} \Subset \Omega$.

(ii) If Ω is a hyperconvex domain in $\mathbb{C}^n$ and $u, f \in \mathcal{E}(\Omega)$ with $u \leq f$ and $h \in \mathcal{F}^a(\Omega)$ then $\max(u, f+h) \in \mathcal{F}^a(\Omega, f)$.

Next, we recall the following.

Definition 2.6. Let $\Omega \subset \mathbb{C}^n$ and $v \in PSH(\Omega)$. v is said to be a maximal plurisubharmonic function if for all compact subset $K \Subset \Omega$ and for every plurisubharmonic function $w \in PSH(\Omega)$, $w \leq v$ on $\Omega \setminus K$ then $w \leq v$ on Ω.
The family of maximal plurisubharmonic functions on Ω is denoted by $MPSH(\Omega)$.

For results concerning to maximal plurisubharmonic functions we refer readers to [17]. In the case $\Omega \subset \mathbb{C}^n$ is a bounded hyperconvex domain and $u \in \mathcal{E}(\Omega)$. Then in [7], Błocki proved that $u \in MPSH(\Omega)$ if and only if it satisfies the homogeneous Monge-Ampère equation $(dd^c u)^n = 0$.

Proposition 2.7. *Let $\Omega \subset \hat{\Omega}$ be bounded hyperconvex domains in $\mathbb{C}^n$. Assume that $u \in \mathcal{F}(\Omega)$ and define*

$$\hat{u} := \sup\{\varphi \in PSH^-(\hat{\Omega}) : \varphi \leq u \ \text{on}\ \Omega\}.$$

If $(dd^c u)^n$ is carried by a pluripolar set of Ω then

$$(dd^c \hat{u})^n = 1_\Omega (dd^c u)^n \ \text{in}\ \hat{\Omega}.$$

Proof. By Lemma 4.5 in [15] we have $\hat{u} \in \mathcal{F}(\hat{\Omega})$ and $(dd^c \hat{u})^n \leq 1_\Omega (dd^c u)^n$ in $\hat{\Omega}$. Since $(dd^c u)^n$ is carried by a pluripolar set of Ω, Theorem 5.11 in [9] implies that $(dd^c u)^n = 1_{\{u=-\infty\}}(dd^c u)^n$ in Ω. Moreover, since $\hat{u} \leq u$ on Ω, by Lemma 4.1 in [1] we get

$$\begin{aligned} 1_\Omega (dd^c u)^n &= 1_{\Omega\cap\{u=-\infty\}}(dd^c u)^n \\ &\leq 1_{\Omega\cap\{\hat{u}=-\infty\}}(dd^c \hat{u})^n \leq (dd^c \hat{u})^n \leq 1_\Omega (dd^c u)^n. \end{aligned}$$

This implies that

$$(dd^c \hat{u})^n = 1_\Omega (dd^c u)^n \text{ in } \hat{\Omega},$$

and the desired conclusion follows. □

Definition 2.8. The Lelong class $\mathcal{L} = \mathcal{L}(\mathbb{C}^n)$ of plurisubharmonic functions in $\mathbb{C}^n$ is given by

$$\mathcal{L} = \mathcal{L}(\mathbb{C}^n) = \{u \in PSH(\mathbb{C}^n) : u(z) \leq log(1+|z|^2) + c_u \quad \text{for } z \in \mathbb{C}^n\}.$$

3 Weak solutions to the complex Monge-Ampère equations.

In this section we prove Theorem 1.1. We need following auxiliary results.

Lemma 3.1. *Let Ω be an open set in $\mathbb{C}^n$ and let $\mathbb{B} \Subset \Omega$ be an open ball. Assume that $u \in \mathcal{D}(\Omega)$ and $v \in \mathcal{F}(\mathbb{B})$ such that*
(i) $\sup_{\mathbb{B}} u < 0$;
(ii) $(dd^c v)^n$ is carried by a pluripolar set $E \subset \mathbb{B}$;
(iii) $(dd^c u)^n$ is carried by a pluripolar set $F \subset \Omega \backslash E$;
(iv) $w := \sup\{\varphi \in PSH(\Omega) : \varphi \leq u$ on Ω and $\varphi \leq v$ on $\mathbb{B}\} \in \mathcal{D}(\Omega)$.
Then, $(dd^c w)^n = (dd^c u)^n + 1_{\mathbb{B}}(dd^c v)^n$ in Ω.

Proof. Without loss of generality we can assume that $E \subset \mathbb{B} \cap \{v = -\infty\}$ and $F \subset (\Omega\backslash E) \cap \{u = -\infty\}$. Let $\{\Omega_j\}$ be an increasing sequence of open sets in $\mathbb{C}^n$ such that $\mathbb{B} = \mathbb{B}(a, r) \Subset \Omega_j \Subset \Omega_{j+1} \Subset \Omega$ and $\Omega = \bigcup_{j=1}^{\infty} \Omega_j$. By the local property of the class $\mathcal{D}(\Omega)$ we may assume that $w \in \mathcal{D}(\Omega_j)$. Set $u_j = u|_{\Omega_j}$. Define

$$w_j := \sup\{\varphi \in PSH(\Omega_j) : \varphi \leq u_j \text{ on } \Omega_j \text{ and } \varphi \leq v \text{ on } \mathbb{B}\}.$$

From $w \leq w_j$ on Ω_j then $w_j \in \mathcal{D}(\Omega_j)$ and $w_j \searrow w$ in Ω. From [11] we have $(dd^c w_j)^n$ converges to $(dd^c w)^n$ in the weak*-topology. Replacing Ω by Ω_j if necessary, we can assume that Ω is bounded and u is a plurisubharmonic function on an open neighborhood of $\overline{\Omega}$. We first prove that

$$(dd^c w)^n = 0 \text{ on } (\{w < u\} \cap \Omega\backslash\overline{\mathbb{B}}) \cup (\{w < \min(u, v)\} \cap \mathbb{B}). \tag{2}$$

Indeed, let $\{v_j\} \subset \mathcal{E}_0(\mathbb{B}) \cap \mathcal{C}(\overline{\mathbb{B}})$ be such that $v_j \searrow v$ in $\mathbb{B}$ and let $\{u_j\} \subset PSH(\Omega) \cap \mathcal{C}(\Omega)$ be such that $u_j \searrow u$ in Ω and $\sup_{\mathbb{B}} u_1 < 0$. We set

$$w_j := \sup\{\varphi \in PSH(\Omega) : \varphi \leq h_j \text{ in } \Omega\},$$

where

$$h_j = \begin{cases} \min(u_j, v_j) & \text{on } \mathbb{B}, \\ u_j & \text{on } \Omega\backslash\mathbb{B}. \end{cases}$$

Since $u_j \leq u_1 < 0 = v_j$ on $\partial\mathbb{B}$ so $h_j \in \mathcal{C}(\Omega)$. Then, $w_j \in PSH(\Omega)$ and $w_j \searrow w$ in Ω. Thanks to Corollary 9.2 in [3] we obtain that $(dd^c w_j)^n = 0$ on $\{w_j < h_j\}$. Let $j \to \infty$, by [11] we get (2).
Next, we claim that

$$(dd^c w)^n = 0 \text{ in } \{u > -\infty\} \cap \{u = w\} \tag{3}$$

Indeed, let $K \subset \{u > -\infty\} \cap \{u = w\}$ be a compact set. Since $K \subset \{w + \frac{1}{j} > u\}$, by Theorem 4.1 in [16] and using the hypotheses we have

$$\begin{aligned} \int_K (dd^c w)^n &= \lim_{j \to +\infty} \int_K (dd^c \max(w + \frac{1}{j}, u))^n \\ &\leq \int_K (dd^c \max(w, u))^n = \int_K (dd^c u)^n. \end{aligned}$$

Moreover, since $(dd^c u)^n$ is carried by a pluripolar set F, we infer

$$\int_K (dd^c w)^n \leq \int_{F \cap \{u > -\infty\}} (dd^c u)^n = 0.$$

Hence, $\int_K (dd^c w)^n = 0$. It follows that

$$(dd^c w)^n = 0 \text{ in } \{u > -\infty\} \cap \{u = w\},$$

and the desired conclusion follows. Similarly, we also have

$$(dd^c w)^n = 0 \text{ in } \mathbb{B} \cap \{v > -\infty\} \cap \{v = w\}.$$

Combining this with (2) and (3) we infer that

$$(dd^c w)^n = 0 \text{ on } (\{u > -\infty\} \cap \Omega \backslash \overline{\mathbb{B}}) \cup (\{\min(u, v) > -\infty\} \cap \mathbb{B}). \tag{4}$$

Now, let $R > 0$ be such that $\Omega \Subset \mathbb{B}(0, R)$ and define

$$\hat{v} := \sup\{\varphi \in PSH^-(\mathbb{B}(0, R)) : \varphi \leq v \text{ on } \mathbb{B}\}.$$

Proposition 2.7 implies that $\hat{v} \in PSH^-(\Omega) \cap \mathcal{D}(\Omega)$ and

$$(dd^c \hat{v})^n = 1_{\mathbb{B}} (dd^c v)^n = 1_E (dd^c v)^n \text{ in } \Omega.$$

We claim that

$$(dd^c w)^n = (dd^c(u + \hat{v}))^n = (dd^c u)^n \text{ on } \{u = -\infty\} \backslash E. \tag{5}$$

Indeed, let $K \subset \{u = -\infty\} \backslash E$ be a compact set. Since $(dd^c \hat{v})^n = 1_E (dd^c v)^n$ and $K \subset \{u = -\infty\} \setminus E$ then

$$\int_K (dd^c \hat{v})^n = 0.$$

Lemma 4.1 and Lemma 4.4 in [1] imply that

$$\begin{aligned}
\int_K (dd^c u)^n &\leq \int_K (dd^c(u + \hat{v}))^n \\
&= \sum_{j=0}^{n} \binom{n}{j} \int_K (dd^c u)^j \wedge (dd^c \hat{v})^{n-j} \\
&\leq \sum_{j=0}^{n} \binom{n}{j} \left(\int_K (dd^c u)^n \right)^{\frac{j}{n}} \left(\int_K (dd^c \hat{v})^n \right)^{\frac{n-j}{n}} \\
&= \int_K (dd^c u)^n.
\end{aligned}$$

It follows that

$$\int_K (dd^c(u + \hat{v}))^n = \int_K (dd^c u)^n.$$

Moreover, since $u + \hat{v} \leq w \leq u$ in Ω, then Theorem 4.1 in [1] implies that

$$\int\limits_K (dd^c u)^n \leq \int\limits_K (dd^c w)^n \leq \int\limits_K (dd^c(u + \hat{v}))^n = \int\limits_K (dd^c u)^n,$$

and the required conclusion follows. Similarly, we also have

$$(dd^c w)^n = (dd^c(u + \hat{v}))^n = (dd^c \hat{v})^n = (dd^c v)^n \text{ on } (\mathbb{B} \cap \{v = -\infty\}) \backslash F.$$

Combining this with (4), (5) and using the hypotheses we infer

$$(dd^c w)^n = (dd^c u)^n + 1_{\mathbb{B}}(dd^c v)^n \text{ in } \Omega\backslash\partial\mathbb{B}. \tag{6}$$

Let $\mathbb{B}'$ be an open ball such that $\mathbb{B} \Subset \mathbb{B}' \Subset \Omega$ and $\sup_{\mathbb{B}'} u < 0$. Put

$$v' := \sup\{\varphi \in PSH^-(\mathbb{B}') : \varphi \leq v \text{ on } \mathbb{B}\}.$$

By Proposition 2.7 we have $v' \in \mathcal{F}(\mathbb{B}')$ and

$$(dd^c v')^n = 1_{\mathbb{B}}(dd^c v)^n \text{ in } \mathbb{B}'.$$

Since $w \leq u < 0$ in $\mathbb{B}'$ and $w \leq v$ in $\mathbb{B}$ so $w \leq v'$ on $\mathbb{B}'$. This implies that

$$w = \sup\{\varphi \in PSH(\Omega) : \varphi \leq u \text{ on } \Omega \text{ and } \varphi \leq v' \text{ on } \mathbb{B}'\}.$$

Applying (6) we get that

$$(dd^c w)^n = (dd^c u)^n + 1_{\mathbb{B}'}(dd^c v')^n \text{ in } \Omega\backslash\partial\mathbb{B}'.$$

Therefore,

$$(dd^c w)^n = (dd^c u)^n + 1_{\mathbb{B}}(dd^c v)^n \text{ in } \Omega.$$

The proof is complete. □

Lemma 3.2. *Let Ω be an open set in $\mathbb{C}^n$ and let $u, v \in \mathcal{D}(\Omega)$ be such that $u \leq v$ in Ω. Assume that $(dd^c v)^n$ is carried by a pluripolar set of Ω and μ is a non-negative Radon measure defined in Ω with $\mu \leq 1_{\{u>-\infty\}}(dd^c u)^n$. Then, for every open ball $\mathbb{B} \Subset \Omega$, there exists $w \in \mathcal{D}(\Omega)$ satisfying*

(i) $u \leq w \leq v$ on Ω;
(ii) $(dd^c w)^n \geq \mu$ in Ω;
(iii) $(dd^c w)^n = \mu + (dd^c v)^n$ in $\mathbb{B}$.

Proof. Fix a ball $\mathbb{B} \Subset \Omega$. Let $\mathbb{B}'$ be an open ball such that $\mathbb{B} \Subset \mathbb{B}' \Subset \Omega$ and $u \in \mathcal{E}(\mathbb{B}')$. Without loss of generality we may assume that $(dd^c v)^n$ is carried by $\{v = -\infty\}$. Define

$$g := (\sup\{\varphi \in PSH^-(\mathbb{B}') : \varphi \leq u \text{ on } \mathbb{B}'\backslash\mathbb{B}\})^*.$$

and

$$f := \sup\{\varphi \in PSH^-(\mathbb{B}') : \varphi \leq \min(g, v) \text{ on } \mathbb{B}'\}.$$

Then, $g \in \mathcal{E}(\mathbb{B}') \cap MPSH(\mathbb{B})$. Since $u \leq v$ in Ω so $u \leq f \leq v$ in $\mathbb{B}'$ and $f = g = u$ on $\mathbb{B}'\backslash(\mathbb{B} \cup E)$ for some pluripolar set $E \subset \partial\mathbb{B}$. By Lemma 4.1 in [1], Corollary 3.1 in [13] and using the hypotheses we have

$$\begin{aligned}(dd^c v)^n = 1_{\{v=-\infty\}}(dd^c v)^n &\leq 1_{\{f=-\infty\}}(dd^c f)^n \\ &\leq (dd^c f)^n \leq (dd^c v)^n + (dd^c g)^n = (dd^c v)^n\end{aligned}$$

on $\mathbb{B}$. It follows that

$$(dd^c f)^n = (dd^c v)^n \text{ on } \mathbb{B}.$$

Now, since the measure $1_{\{u>-\infty\}}(dd^c u)^n$ vanishes on all pluripolar sets of Ω and $\mu \leq 1_{\{u>-\infty\}}(dd^c u)^n$ in Ω so by Lemma 5.14 in [9] there exists $h \in \mathcal{F}^a(\mathbb{B})$ such that

$$(dd^c h)^n = \mu \text{ in } \mathbb{B}.$$

It is easy to see that $\max(u, h+f) \in \mathcal{F}^a(\mathbb{B}, f)$ and

$$(dd^c f)^n \leq \mu + (dd^c v)^n \leq (dd^c \max(u, h+f))^n \text{ in } \mathbb{B}.$$

Thanks to Lemma 4.1 in [14] there is $\psi \in \mathcal{F}^a(\mathbb{B}, f)$ such that $u \leq \psi \leq f$ and

$$(dd^c \psi)^n = \mu + (dd^c v)^n \text{ in } \mathbb{B}.$$

Let w be the smallest plurisubharmonic majorant of the function

$$\eta = \begin{cases} \psi & \text{in } \mathbb{B}, \\ u & \text{in } \Omega \backslash \mathbb{B}. \end{cases}$$

Since $u \leq \psi \leq f \leq g$ on $\mathbb{B}$, we have $w \in \mathcal{D}(\Omega)$ and $u \leq w \leq v$ in Ω. It is easy to see that

$$(dd^c w)^n \geq \mu \text{ in } \Omega \backslash \mathbb{B} \tag{7}$$

and

$$(dd^c w)^n = \mu + (dd^c v)^n \text{ on } \mathbb{B}. \tag{8}$$

Indeed, by the definition of w we note that $w = u$ in the interior of $\Omega \setminus \mathbb{B}$ and $w = \psi$ in $\mathbb{B}$. Hence, we have $(dd^c w)^n \geq \mu$ on the interior of $\Omega \setminus \mathbb{B}$ and (8) holds. In order to prove $(dd^c w)^n \geq \mu$ on $\Omega \setminus \mathbb{B}$ it suffices to prove $(dd^c w)^n \geq \mu$ on $\partial\mathbb{B}$. By the definition of w it follows that $w = u$ on $\partial\mathbb{B} \setminus E$, where E is a pluripolar subset of $\partial\mathbb{B}$ containing $\{u = -\infty\}$. Let $K \subset \partial\mathbb{B} \backslash E$ be a compact set. Since $K \subset \{u + \frac{1}{j} > w\}$, by Theorem 4.1 in [16] we have

$$\begin{aligned} \mu(K) &\leq \int_K (dd^c u)^n = \lim_{j \to +\infty} \int_K (dd^c \max(u + \frac{1}{j}, w))^n \\ &\leq \int_K (dd^c \max(u, w))^n = \int_K (dd^c w)^n. \end{aligned}$$

It follows that

$$(dd^c w)^n \geq \mu \text{ on } \partial\mathbb{B} \backslash E.$$

Hence,

$$(dd^c w)^n \geq \mu \text{ on } \partial\mathbb{B}.$$

Combining this with (7) and (8) we obtain

$$(dd^c w)^n \geq \mu \text{ in } \Omega.$$

The proof is complete. $\square$

Proof of Theorem 1.1. Let $\{\mathbb{B}_j\}$ be a sequence of open balls such that $\mathbb{B}_j \Subset \Omega$ and $\Omega = \bigcup_{j=1}^{\infty} \mathbb{B}_j$. Put $\mathbb{B}_0 := \emptyset$. We first claim that there exists a decreasing sequence $\{f_j\} \subset \mathcal{D}(\Omega)$ such that $u \leq f_j \leq v$ and

$$(dd^c f_j)^n = 1_{(\bigcup_{k=0}^{j-1} \mathbb{B}_k) \cap \{u=-\infty\}} \mu \text{ on } \Omega.$$

Indeed, let $f_1 := v$. Then, $f_1 \in \mathcal{D}(\Omega)$ and $(dd^c f_1)^n = 0$ in Ω. Let $j \geq 1$ be an integer number. Assume by induction that we have determined f_j. We find f_{j+1} as follows. Let $\mathbb{B}'_j$ be an open ball such that $\mathbb{B}_j \Subset \mathbb{B}'_j \Subset \Omega$. By Theorem 4.14 in [1], there exists $v_j \in \mathcal{F}(\mathbb{B}'_j)$ such that $v_j \geq u$ and

$$(dd^c v_j)^n = 1_{(\mathbb{B}_j \setminus \bigcup_{k=0}^{j-1} \mathbb{B}_k) \cap \{u=-\infty\}} \mu \text{ on } \mathbb{B}'_j.$$

Set $c_j := \sup_{\mathbb{B}'_j} v + \lambda_j$ where λ_j is choosen such that $c_j \geq 0$. Next, we define

$$f_{j+1} = \sup\{\varphi \in PSH(\Omega) : \varphi \leq f_j - c_j \text{ in } \Omega \text{ and } \varphi \leq v_j \text{ in } \mathbb{B}'_j\} + c_j.$$

Then we have $u \leq f_{j+1} \leq f_j \leq v$ on Ω and by Theorem 1.2 in [6] we have $f_{j+1} \in \mathcal{D}(\Omega)$. Lemma 3.1 implies that

$$(dd^c f_{j+1})^n = (dd^c f_j)^n + 1_{\mathbb{B}'_j} (dd^c v_j)^n = 1_{(\bigcup_{k=0}^{j} \mathbb{B}_k) \cap \{u=-\infty\}} \mu \text{ in } \Omega.$$

This proves the claim. Put $f := \lim\limits_{j \to \infty} f_j$ in Ω. Then, $f \in PSH(\Omega)$ and $u \leq f \leq v$ in Ω. By Theorem 1.2 in [6] and using the main result in [11] we infer that $f \in \mathcal{D}(\Omega)$ and

$$(dd^c f)^n = 1_{\{u=-\infty\}} \mu \text{ in } \Omega.$$

We now set

$$w := \sup\{\varphi \in \mathcal{D}(\Omega) : \varphi \leq f \text{ and } (dd^c \varphi)^n \geq 1_{\{u>-\infty\}} \mu\}.$$

Because $u \leq w \leq f \leq v$ in Ω, by Theorem 1.2 in [6] we have $w \in \mathcal{D}(\Omega)$. Since the measure $1_{\{u>-\infty\}} \mu$ vanishes on pluripolar sets of Ω, by Proposition 4.3 in [16] and using the Choquet lemma we can choose a increasing sequence $\{\varphi_j\} \subset \mathcal{D}(\Omega)$ such that $\varphi_j \nearrow w$ a.e. in Ω and

$$(dd^c \varphi_j)^n \geq 1_{\{u>-\infty\}} \mu \text{ in } \Omega.$$

By the main result in [11] we note that $(dd^c \varphi_j)^n \to (dd^c w)^n$ weakly in Ω, and hence,

$$(dd^c w)^n \geq 1_{\{u>-\infty\}} \mu \text{ in } \Omega.$$

Let $\mathbb{B} \Subset \Omega$ be an open ball. By Lemma 3.2 there exists $\psi \in \mathcal{D}(\Omega)$ such that
(i) $w \leq \psi \leq f$ on Ω;
(ii) $(dd^c \psi)^n \geq 1_{\{u>-\infty\}} \mu$ in Ω;
(iii) $(dd^c \psi)^n = 1_{\{u>-\infty\}} \mu + (dd^c f)^n$ in $\mathbb{B}$.
From the definition of w it is easy to see that $w = \psi$ in Ω, and therefore, by (iii) we have

$$(dd^c w)^n = 1_{\{u>-\infty\}} \mu + (dd^c f)^n = \mu \text{ in } \mathbb{B},$$

and the desired conclusion follows. □

4 Global solutions of the complex Monge-Ampère equation in the class $\mathcal{D}(\mathbb{C}^n)$.

In this section as an application of Theorem 1.1, we study global solutions of the complex Monge-Ampère equation in the class $\mathcal{D}(\mathbb{C}^n)$. Namely, we prove the following.

Theorem 4.1. *Let $u \in \mathcal{D}(\mathbb{C}^n)$, $(dd^c u)^n(\{0\}) = (2\pi)^n$, $u(z) \le log|z|$ for $z \in \mathbb{C}^n$. Assume that μ is a non-negative Radon measure in $\mathbb{C}^n$, $\mu(\{0\}) = (2\pi)^n$ and $\mu \le (dd^c u)^n$. Then there exists $w \in \mathcal{D}(\mathbb{C}^n)$, $u \le w \le log|z|$ for $z \in \mathbb{C}^n$ such that $(dd^c w)^n = \mu$.*

Proof. Set $\Omega = \mathbb{C}^n \setminus \{0\}, v(z) = log|z|$, $z \in \Omega$. Then $v \in MPSH(\Omega) \cap L^\infty_{loc}(\Omega)$ (see [17]). Hence, $v \in \mathcal{D}(\Omega)$. At the same time, we have $u|_\Omega \le v$ on Ω. From [6] we note that $u \in \mathcal{D}(\Omega)$. Put $\mu_1 = \mu|_\Omega$. By the hypothesis, $\mu_1 \le (dd^c u)^n$ on Ω. Theorem 1.1 implies that there exists $h \in \mathcal{D}(\Omega), u \le h \le v$ on Ω such that

$$(dd^c h)^n = \mu_1 = \mu|_{\mathbb{C}^n \setminus \{0\}}.$$

Since $u \le h \le v = log|z|$ on Ω then we have $\lim\limits_{y \to 0, y \neq 0} h(y) = -\infty$. Set

$$w(z) = \begin{cases} h(z) & \text{if } z \neq 0 \\ -\infty & \text{if } z = 0. \end{cases}$$

Then $w \in PSH(\mathbb{C}^n)$ and we have $u(z) \le w(z) \le log|z|$ for all $z \in \mathbb{C}^n$. By [6] this yields that $w \in \mathcal{D}(\mathbb{C}^n)$. Moreover, we may assume that $u, w \in \mathcal{E}(\mathbb{B}(0,1))$ where $\mathbb{B}(0,1)$ is the unit ball in $\mathbb{C}^n$. By Lemma 4.1 in [1] it follows that

$$(2\pi)^n = \int\limits_{\{0\}} (dd^c log|z|)^n \le \int\limits_{\{0\}} (dd^c w)^n \le \int\limits_{\{0\}} (dd^c u)^n = (2\pi)^n.$$

Hence, $\int\limits_{\{0\}} (dd^c w)^n = (2\pi)^n$. Now we have

$$\begin{aligned}(dd^c w)^n = (dd^c w)^n|_{\mathbb{C}^n \setminus \{0\}} + (dd^c w)^n(\{0\}) &= (dd^c h)^n|_{\mathbb{C}^n \setminus \{0\}} + (2\pi)^n \\ &= \mu|_{\mathbb{C}^n \setminus \{0\}} + (2\pi)^n = \mu.\end{aligned}$$

The proof is complete. □

From Theorem 4.1 we obtain a global solution in the class $\mathcal{D}(\mathbb{C}^n) \cap \mathcal{L}$ as follows.

Corollary 4.2. *Under the hypotheses of Theorem 4.1 there exists $w \in \mathcal{D}(\mathbb{C}^n) \cap \mathcal{L}$ such that $(dd^c w)^n = \mu$.*

Proof. Indeed, it is easy to see that $log|z| \le log(1 + |z|^2) - log2$ for $z \in \mathbb{C}^n$ and the desired conclusion follows. □

Next, we give the following result.

Theorem 4.3. *Let μ be a non-negative Borel measure on $\mathbb{C}^n$ such that $\mu \le (dd^c log(1 + |z|^2))^n$ on $\mathbb{C}^n$. Assume that there exists $r \ge 2$ such that $\mu(|z| = r) = (dd^c log(1 + |z|^2))^n(|z| = r)$. Then there exists $w \in PSH(\mathbb{C}^n) \cap \mathcal{D}(\mathbb{C}^n)$ such that $(dd^c w)^n = \mu$.*

Proof. Without loss of generality we give the proof of Theorem 4.3 for the case $r = 2$. For other cases the proof is similar. First, we choose $c > 0$ such that $3\log|z| \geq log(1+|z|^2) + c$ for $|z| \geq 2$ and

$$\lim_{\substack{x\to z\\|x|>2}} log(1+|x|^2) + c = 3\lim_{\substack{x\to z\\|x|>2}} log|x| = log8,$$

at $|z| = 2$. Then $\mu \leq (dd^c(log(1+|z|^2)+c))^n$ and $log(1+|z|^2)+c \leq 3log|z|$ for $|z| > 2$. By Theorem 1.1 there exists $u_1 \in \mathcal{D}(|z| > 2)$ such that $(dd^c u_1)^n = \mu$ and $log(1+|z|^2)+c \leq u_1 \leq 3log|z|$ on $|z| > 2$. Let $\varphi(z) = log(1+|z|^2)+c, |z| = 2$. Then $\varphi \in C(|z| = 2)$. In the strictly pseudoconvex domain $\mathbb{B}(0,2) = \{z \in \mathbb{C}^n : |z| < 2\}$ consider the Dirichlet problem:

$$\begin{cases} u \in PSH \cap L^\infty(\mathbb{B}(0,2)),\\ (dd^c u)^n = \mu \text{ in } \mathbb{B}(0,2),\\ \lim\limits_{\substack{x\to z\\|x|<2}} u(x) = \varphi(z), |z| = 2. \end{cases} \tag{9}$$

Since there exists a subsolution $v(z) = log(1+|z|^2)+c$, $\mu \leq (dd^c v)^n$, $\lim\limits_{x\to z} v(x) = \varphi(z), |z| = 2$ then by Theorem A in [18] we can find $u_2 \in PSH \cap L^\infty(\mathbb{B}(0,2))$ such that $(dd^c u_2)^n = \mu$ on $\mathbb{B}(0,2)$ and $\lim\limits_{\substack{x\to z\\|x|<2}} u_2(x) = \varphi(z), |z| = 2$. Note that we have $\lim\limits_{\substack{x\to z\\|x|<2}} u_2(x) = \lim\limits_{\substack{x\to z\\|x|>2}} u_1(x) = \varphi(z) = log(1+|z|^2)+c, |z| = 2$. Moreover, by the comparison principle in [17] we have $u_2(z) \geq log(1+|z|^2)+c$ in $\mathbb{B}(0,2)$. Now set

$$w(z) = \begin{cases} u_2(z) & \text{if } |z| < 2,\\ \varphi(z) = log(1+|z|^2)+c & \text{if } |z| = 2,\\ u_1(z) & \text{if } |z| > 2. \end{cases} \tag{10}$$

Then $w \in PSH(\mathbb{C}^n) \cap L^\infty_{loc}(\mathbb{C}^n)$ and, hence, $w \in \mathcal{D}(\mathbb{C}^n)$. It remains to prove $(dd^c w)^n = \mu$. It is clear that $(dd^c w)^n = \mu$ on $\{|z| < 2\}$ and $\{|z| > 2\}$. Hence, it remains to show that $(dd^c w)^n(|z| = 2) = \mu(|z| = 2)$. Let $K \Subset \{|z| = 2\}$ be arbitrary. Then for every $j \geq 1$, we have $K \subset \{w + \frac{1}{j} > log(1+|z|^2)+c\}$. Proposition 4.2 in [4] implies that

$$\begin{aligned} (dd^c w)(K) &= \int_K ((dd^c(w+\frac{1}{j}))^n = \lim_{j\to\infty}\int_K (dd^c \max(w+\frac{1}{j}, log(1+|z|^2)+c))^n\\ &\leq \int_K (dd^c \max(w, log(1+|z|^2)+c))^n = \int_K (dd^c log(1+|z|^2))^n\\ &= (dd^c log(1+|z|^2))^n(K). \end{aligned}$$

Hence, $(dd^c w)^n(K) \leq (dd^c log(1+|z|^2))^n(K)$ for all $K \Subset \{|z| = 2\}$. This yields that

$$(dd^c w)^n(|z| = 2) \leq (dd^c log(1+|z|^2))^n(|z| = 2) = \mu(|z| = 2). \tag{11}$$

On the other hand, for all $K \Subset \{|z| = 2\}$, similarly as above, we have $K \subset \{log(1+|z|^2)+c+\frac{1}{j} > w\}$ for $j \geq 1$. Once more, using Proposition 4.2 in [4] we get that

$$\mu(K) \leq \int_K (dd^c(log(1+|z|^2)+c))^n = \lim_{j\to\infty} \int_K (dd^c \max(log(1+|z|^2)+c+\frac{1}{j}, w))^n$$
$$\leq \int_K (dd^c \max(log(1+|z|^2)+c, w))^n = \int_K (dd^c w)^n = (dd^c w)^n(K).$$

Hence,

$$\mu(|z|=2) \leq (dd^c w)^n(|z|=2). \tag{12}$$

Coupling (11) and (12) we get that $\mu(|z|=2) = (dd^c w)^n(|z|=2)$ and the desired conclusion follows. The proof is complete.

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□

References

[1] P. Åhag, U. Cegrell, R. Czyż and P. H. Hiep, *Monge-Ampère measures on pluripolar sets*, J. Math. Pures Appl., **92** (2009), 613–627.

[2] E. Bedford and B. A. Taylor, *The Dirichlet problem for a complex Monge-Ampère operator*, Invent. Math., **37** (1976), 1–44.

[3] E. Bedford and B. A. Taylor, *A new capacity for plurisubharmonic functions.* Acta Math., **149** (1982), 1–41.

[4] E. Bedford and B. A. Taylor, *Fine Topology, Šilov Boundary, and $(dd^c)^n$*, J. Funct. Anal., **72**(1987), 225–251.

[5] Z. Błocki, *On the definition of the Monge-Ampère operator in $\mathbb{C}^2$*, Math. Ann **328** (2004), 415–423.

[6] Z. Błocki, *The domain of definition of the complex Monge-Ampère operator*, Amer. J. Math., **128** (2006), 519–530.

[7] Z. Błocki, *A note on maximal plurisubharmonic functions*, Uzbek. Math. J., 2009, no 1, 28–32.

[8] U. Cegrell, *Pluricomplex energy*, Acta Math., **180** (1998), 187–217.

[9] U. Cegrell, *The general definition of the complex Monge-Ampère operator*, Ann. Inst. Fourier., **54** (2004), 159–179.

[10] U. Cegrell, *A general Dirichlet problem for the complex Monge-Ampère operator*, Ann. Polon. Math., **94** (2008), 131–147.

[11] U. Cegrell, *Convergence in capacity*, Canad. Math. Bull., **55** (2012), 242–248.

[12] L. M. Hai, T. V. Thuy and N. X. Hong, *A note on maximal subextensions of plurisubharmonic functions*, Acta Math Vietnam, **43**(2018), 137–146, http://doi.org/10. 1007/s40306-017-0234-z.

[13] N. X. Hong, *Monge-Ampère measures of maximal subextensions of plurisubharmonic functions with given boundary values*, Complex Var. Elliptic Equ., **60** (2015), no. 3, 429–435.

[14] N. X. Hong, N. V. Trao and T. V. Thuy, *Convergence in capacity of plurisubharmonic functions with given boundary values*, Inter. J. Math, Vol. 28, No. 3 (2017) 1750018 (14 pages).

[15] P. H. Hiep, *Pluripolar sets and the subextension in Cegrell's classes*, Complex Var. Elliptic Equ., **53**.7 (2008), 675–684.

[16] N. V. Khue and P. H. Hiep, *A comparison principle for the complex Monge-Ampère operator in Cegrell's classes and applications*, Trans. Am. Math. Soc., **361** (2009), 5539–5554.

[17] M. Klimek, Pluripotential Theory, The Clarendon Press Oxford University Press, New York., 1991, Oxford Science Publications.

[18] S. Kołodziej, *The range of the complex Monge-Ampère operator, II*, Indiana U. Math. J., **44** (1995), 765–782.

[19] Y. T. Siu, *Extension of meromorphic maps into Kähler manifolds*, Ann. of Math. (2) 102(1975), 421–462.