

RIGIDITY PROPERTIES OF P -BIHARMONIC MAPS AND P -BIHARMONIC SUBMANIFOLDS

W. BARKER, N. T. DUNG, K. SEO* AND N. D. TUYEN†

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Abstract

We give some rigidity properties of a p -biharmonic map $u : (M, g) \rightarrow (N, h)$ between Riemannian manifolds (M^n, g) and (N^m, h) . We first provide various sufficient conditions for p -biharmonic maps to be harmonic. Moreover, when the map u is an isometric immersion, by assuming that the $L^{\frac{n}{2}}$ -norm of the sectional curvature on M is sufficiently small or if the fundamental tone of the p -biharmonic submanifold is sufficiently big, it is proved that M is minimal.

1 Introduction and results

Let $u : (M^n, g) \rightarrow (N^m, h)$ be a smooth map between Riemannian manifolds (M^n, g) and (N^m, h) . The differential du can be considered as a section of the vector bundle $T^*M \otimes u^{-1}TN$. Given a local orthonormal frame $\{e_i\}$ on M , $|du|$ can be computed as

$$|du|^2 = \sum_{i=1}^n \langle du(e_i), du(e_i) \rangle,$$

where $|du|(x)$ is the Hilbert-Schmidt norm of $(du)(x)$ induced by the metrics g and h at a given point $x \in M$. If the map u is a critical point of the following energy functional

$$E(u) = \frac{1}{2} \int_M |du|^2$$

then we call the map u *harmonic*. It is well-known that the Euler-Lagrange equation [CL16] of the energy E is given by

$$\tau(u) := \sum_{i=1}^n \left[\tilde{\nabla}_{e_i} du(e_i) - du(\nabla_{e_i} e_i) \right] = 0,$$

where $\tilde{\nabla}$ is the Levi-Civita connection on the pullback bundle $u^{-1}TN$ and $\tau(u)$ is called the *tension field* of u .

*Corresponding author

†Corresponding author

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On the other hand, in the study of higher-order elliptic problems, it is natural to consider the biharmonic maps which is the critical point of the bienergy functional

$$E_2(u) = \frac{1}{2} \int_M |\tau(u)|^2.$$

We note that the Euler-Lagrange equation [CL16] of $E_2(u)$ is given by

$$\tau_2(u) := \Delta \tau(u) - \sum_{i=1}^n R^N(\tau(u), du(e_i)) du(e_i) = 0.$$

In a general way, Hornung-Moser [HM14] (see also [HF14]) considered the p -bienergy ($p > 1$) functional as follows:

$$E_p(u) = \int_M |\tau(u)|^p.$$

The p -bitension field $\tau_p(u)$ is defined by

$$\tau_p(u) := \Delta(|\tau(u)|^{p-2} \tau(u)) - \sum_{i=1}^n R^N(|\tau(u)|^{p-2} \tau(u), du(e_i)) du(e_i).$$

The Euler-Lagrange equation for $E_p(u)$ is given by $\tau_p(u) = 0$ and a map u satisfying that $\tau_p(u) = 0$ is called a p -biharmonic map. One of the most interesting problems in the biharmonic theory is the following problem, which was proposed by Chen in 1988:

Conjecture (Chen's conjecture). Any biharmonic submanifold in Euclidean space $\mathbb{R}^n$ is minimal.

More generally, Caddeo-Montaldo-Oniciuc [CMO01] proposed the generalized Chen's conjecture as follows.

Conjecture (Generalized Chen's conjecture). Any biharmonic submanifold in a Riemannian manifold with nonpositive sectional curvature is minimal.

Chen's and the generalized Chen's conjectures have been intensively studied. For example, Chen [Chen91] and Jiang [Jia87] showed that Chen's conjecture is true for biharmonic surfaces in $\mathbb{R}^3$. Hasanis-Vlachos [HV95] and Fu-Hong-Zhan [FHZ21] gave an affirmative answer to Chen's conjecture in $\mathbb{R}^4$ and $\mathbb{R}^5$, respectively. Moreover, Ou-Tang [OT12] showed that the generalized Chen's conjecture is false. However it is interesting to find sufficient conditions for biharmonic submanifolds to be minimal. Nakauchi-Urakawa [NU11, NU13] proved the generalized Chen's conjecture when the L^2 -norm of the mean curvature vector is finite. Motivated by this result, Luo [Luo15] extended their result under assumption on finiteness of the L^p -norm of the mean curvature vector for some $0 < p < \infty$. Furthermore, Nakauchi-Urakawa-Gudmundsson [NUG14] showed that the map is harmonic if the energy and bienergy of the domain manifold are finite and if the curvature of the target manifold is nonpositive. Recently, Seo-Yun [SY22] studied biharmonic maps and biharmonic submanifolds with small curvature integral. By assuming the domain manifold satisfies a Sobolev inequality, they gave many sufficient conditions for biharmonic maps to be harmonic and for biharmonic submanifolds to be minimal. We refer the readers to [AM13, BMO10, CMO01, CMO01b, CMO02, Chen91, Chen96, CM13, Def98, DIM92, Fu14, Jia86, Luo14, MAE14, NUG14, ONI02, OU10] for further discussion in this field. On the other direction, motivated by Chen's conjecture, Han [Han15] proposed the following conjecture for p -biharmonic submanifolds.

Conjecture. Every complete p -biharmonic submanifolds in non-positively curved Riemannian manifold is minimal.

By using the method developed in [Luo15], Han [Han15] proved several results on the nonexistence of p -biharmonic submanifolds. Cao-Luo [CL16] studied the nonexistence result for general p -biharmonic submanifolds. Moreover, Han-Zhang [HZ15] investigated p -biharmonic maps and obtained the harmonicity of the biharmonic maps. They also obtained that any weakly convex p -biharmonic hypersurfaces in space form $N(c)$ with $c \leq 0$ is minimal. Inspired by these investigations, our aim in this paper is to study p -biharmonic maps and p -biharmonic submanifolds with small curvature integral. Firstly, if $L^{\frac{n}{2}}$ -norm of $|R^N \circ u| \cdot |du|^2$ is sufficiently small, then we obtain the harmonicity of the p -biharmonic map as follows.

THEOREM 1.1. *Let $u : (M^n, g) \rightarrow (N^m, h)$ be a p -biharmonic map from a complete noncompact Riemannian manifold (M^n, g) into a Riemannian manifold (N^m, h) with $\int_M |\tau(u)|^Q < \infty$ for some constant $Q > p - 1$. Assume that (M^n, g) satisfies the Sobolev inequality (3.1) and*

$$\| |R^N \circ u| \cdot |du|^2 \|_{L^{\frac{n}{2}}(M)} := \left(\int_M (|R^N \circ u| \cdot |du|^2)^{\frac{n}{2}} \right)^{\frac{2}{n}} < \frac{4(p-1)(Q+1-p)}{Q^2 C_s},$$

where C_s denotes the Sobolev constant. Then u is harmonic.

We note that Theorem 1.1 recovers Theorem 2.2 and Theorem 2.3 in [SY22] (see Corollaries 3.1 and 3.3). When the product of L^n -norm of $|du|^2$ and L^n -norm of $|R^N \circ u|$ is sufficiently small, we get the harmonicity of the p -biharmonic map as follows.

THEOREM 1.2. *Let $u : (M^n, g) \rightarrow (N^m, h)$ be a p -biharmonic map from a complete noncompact Riemannian manifold (M^n, g) into a Riemannian manifold (N^m, h) with $\int_M |\tau(u)|^Q < \infty$ for some constant $Q > p - 1$. Assume that (M^n, g) satisfies the Sobolev inequality (3.1) and*

$$\| |R^N \circ u| \|_{L^n(M)} \| |du|^2 \|_{L^n(M)} < \frac{4(p-1)(Q+1-p)}{Q^2 C_s},$$

where C_s denotes the Sobolev constant. Then u is harmonic.

Let Σ be a complete noncompact Riemannian manifold. Denote by $\lambda_1(\Omega)$ the first positive eigenvalue of the following eigenvalue problem for a bounded domain $\Omega \subset \Sigma$

$$\begin{cases} \Delta f + \lambda f = 0 & \text{in } \Omega, \\ f = 0 & \text{on } \partial\Omega, \end{cases}$$

where Δ denotes the Laplace-Beltrami operator on Σ . Then the *fundamental tone* $\lambda_1(\Sigma)$ is defined by

$$\lambda_1(\Sigma) := \inf_{\Omega} \lambda_1(\Omega),$$

where the infimum is taken over all bounded domains in Σ . Replacing the condition on the Sobolev inequality by the condition on the fundamental tone of the domain manifold, we obtain a similar result as follows.

THEOREM 1.3. *Let $u : (M^n, g) \rightarrow (N^m, h)$ be a p -biharmonic map from a complete noncompact Riemannian manifold (M^n, g) into a Riemannian manifold (N^m, h) with $|R^N \circ u| \cdot |du|^2 \leq K$ for some constant $K > 0$ and $\int_M |\tau(u)|^Q < \infty$ for some constant $Q > p - 1$. Assume that the fundamental tone of M satisfies $\lambda_1(M) > \frac{Q^2 K}{4(p-1)(Q+1-p)}$. Then u is harmonic.*

In particular, if $|du| \leq C$, $|R^N \circ u| \leq D$ and $p = Q = 2$, then we are able to obtain Theorem 2.4 in [SY22] (see Corollary 3.6). On the other hand, when the Weyl curvature tensor of the target manifold $W^N = 0$, we have the following result.

THEOREM 1.4. Let $u : (M^n, g) \rightarrow (N^m, h)$ be a p -biharmonic map from a complete noncompact Riemannian manifold (M^n, g) into a Riemannian manifold (N^m, h) satisfying that $W^N = 0, S^N \leq 0$ and $\int_M |\tau(u)|^Q < \infty$ for some constant $Q > p - 1$. Assume that (M^n, g) satisfies the Sobolev inequality (3.1) and

$$\|Z^N \circ u \cdot |du|^2\|_{L^{\frac{n}{2}}(M)} < \frac{(m-2)(p-1)(Q+1-p)}{Q^2 C_s},$$

where C_s denotes the Sobolev constant. Then u is harmonic.

In the same way as Theorem 1.2, we have the following theorem.

THEOREM 1.5. Let $u : (M^n, g) \rightarrow (N^m, h)$ be a p -biharmonic map from a complete noncompact Riemannian manifold (M^n, g) into a Riemannian manifold (N^m, h) satisfying that $W^N = 0, S^N \leq 0$ and $\int_M |\tau(u)|^Q < \infty$ for some constant $Q > p - 1$. Assume that (M^n, g) satisfies the Sobolev inequality (3.1) and

$$\|Z^N \circ u\|_{L^n(M)} \|du\|_{L^n(M)} < \frac{(m-2)(p-1)(Q+1-p)}{Q^2 C_s},$$

where C_s denotes the Sobolev constant. Then u is harmonic.

Replacing the Sobolev inequality condition by a certain condition on the fundamental tone of the domain manifold M , we obtain a rigidity result for p -biharmonic as follows.

THEOREM 1.6. Let $u : (M^n, g) \rightarrow (N^m, h)$ be a p -biharmonic map from a complete noncompact Riemannian manifold (M^n, g) into a Riemannian manifold (N^m, h) satisfying that $W^N = 0, S^N \leq 0$, $|Z^N \circ u| \cdot |du|^2 \leq K$ for some constant $K > 0$ and $\int_M |\tau(u)|^Q < \infty$ for some constant $Q > p - 1$. Assume that $\lambda_1(M) > \frac{Q^2 K}{(m-2)(p-1)(Q+1-p)}$. Then u is harmonic.

Recall that, if $u : (M^n, g) \rightarrow (N^m, h)$ is a p -biharmonic isometric immersion, then the map u is called p -biharmonic. Moreover, any 2-biharmonic submanifolds are called *biharmonic* simply. Using some condition on small integral of curvature and assuming that the Sobolev inequality (3.1) holds on M , we are able to prove the following rigidity result.

THEOREM 1.7. Let $u : (M^n, g) \rightarrow (N^m, h)$ be a p -biharmonic isometric immersion of a complete noncompact submanifold M into a Riemannian manifold N satisfying that $\int_M |\vec{H}|^Q < \infty$ for some constant $Q > p - 1$, where $\vec{H}$ denotes the mean curvature vector field. Assume that (M^n, g) satisfies the Sobolev inequality (3.1) and

$$\|R^N \circ u\|_{L^{\frac{n}{2}}(M)} < \frac{4(p-1)(Q+1-p)}{Q^2 C_s},$$

where C_s denotes the Sobolev constant. Then u is minimal.

It should be mentioned that our approach is slight different from [SY22], where Seo-Yun considered two cases $m = n + 1$ and $m > n + 1$ separately. However, our approach here is applicable to both cases. Instead of using the Sobolev inequality, we obtain a similar result by using the fundamental tone of the domain manifold as follows.

THEOREM 1.8. Let $u : (M^n, g) \rightarrow (N^m, h)$ be a p -biharmonic isometric immersion of a complete noncompact submanifold M into a Riemannian manifold N satisfying that $|R^N \circ u| \leq K$ for some constant $K > 0$, and $\int_M |\vec{H}|^Q < \infty$ for some constant $Q > p - 1$, where $\vec{H}$ denotes the mean curvature vector field. Assume that the fundamental tone of M satisfies $\lambda_1(M) > \frac{Q^2 K}{4(p-1)(Q+1-p)}$. Then u is minimal.

Finally, if the target manifold (N^{n+1}, h) is Einstein, i.e., $\text{Ric}^N = \frac{\text{scal}^N}{n+1} h$, then we obtain the following rigidity property.

THEOREM 1.9. *Let $u : (M^n, g) \rightarrow (N^{n+1}, h)$ be a p -biharmonic isometric immersion of a complete noncompact hypersurface M into an Einstein manifold N with nonnegative constant scalar curvature S^N . Assume that $\int_M |\vec{H}|^Q < \infty$ for some constant $Q > p - 1$ and $\lambda_1(M) > \frac{S^N Q^2}{4(n+1)(p-1)(Q+1-p)}$, where $\vec{H}$ denotes the mean curvature vector field. Then u is minimal.*

The rest of this paper is organized as follows: In Section 2, we recall some preliminary background of p -biharmonic maps. In Section 3, we prove many results for p -biharmonic maps with small curvature integral. Finally, the rigidity results of p -biharmonic submanifolds are proved in Section 4.

2 Preliminaries

Let $u : (M^n, g) \rightarrow (N^m, h)$ be a p -biharmonic map between Riemannian manifolds (M^n, g) and (N^m, h) . Then the Bochner-Weitzenböck formula for $\tau(u)$ is given by

$$\frac{1}{2} \Delta |\tau(u)|^2 = |\nabla \tau(u)|^2 + \langle \Delta \tau(u), \tau(u) \rangle.$$

Therefore

$$\frac{1}{2} \Delta |\tau(u)|^{2(p-1)} = |\nabla (|\tau(u)|^{p-2} \tau(u))|^2 + \langle \Delta (|\tau(u)|^{p-2} \tau(u)), |\tau(u)|^{p-2} \tau(u) \rangle.$$

Since $\tau_p(u) = 0$, we have

$$\begin{aligned} \frac{1}{2} \Delta |\tau(u)|^{2(p-1)} &= |\nabla (|\tau(u)|^{p-2} \tau(u))|^2 \\ (2.1) \quad &+ \left\langle \sum_{i=1}^n R^N (|\tau(u)|^{p-2} \tau(u), du(e_i)) du(e_i), |\tau(u)|^{p-2} \tau(u) \right\rangle. \end{aligned}$$

By the fact that $\Delta(fg) = f\Delta g + g\Delta f + 2\langle \nabla f, \nabla g \rangle$ and the Kato inequality, the above inequality yields

$$\begin{aligned} |\tau(u)|^{p-1} \Delta |\tau(u)|^{p-1} &\geq \left\langle \sum_{i=1}^n R^N (|\tau(u)|^{p-2} \tau(u), du(e_i)) du(e_i), |\tau(u)|^{p-2} \tau(u) \right\rangle \\ &\geq -|R^N \circ u| |du|^2 |\tau(u)|^{2(p-1)}. \end{aligned}$$

Therefore we have

$$|\tau(u)| \Delta |\tau(u)|^{p-1} \geq -|R^N \circ u| |du|^2 |\tau(u)|^p.$$

Fix a point $x_0 \in M$. Denote $r(x)$ by the geodesic distance on M from x_0 to x . Choose $\varphi \in C_0^\infty(M)$ satisfying for $r > 0$,

$$(2.2) \quad \varphi(x) = \begin{cases} 1, & \text{if } r(x) \leq r \\ \in [0, 1] \text{ and } |\nabla \varphi|(x) \leq \frac{2}{r}, & \text{if } r < r(x) \leq 2r \\ 0, & \text{if } r(x) > 2r. \end{cases}$$

The above inequality yields

$$\int_M \varphi^2 |\tau(u)|^{q+1} \Delta |\tau(u)|^{p-1} \geq - \int_M \varphi^2 |R^N \circ u| |du|^2 |\tau(u)|^{p+q}.$$

Applying integration by parts for the term in the left hand side gives

$$\int_M \varphi^2 |\tau(u)|^{q+1} \Delta |\tau(u)|^{p-1} = - \int_M \langle \nabla(\varphi^2 |\tau(u)|^{q+1}), \nabla |\tau(u)|^{p-1} \rangle.$$

Therefore the above inequality yields

$$\begin{aligned} \int_M \varphi^2 |R^N \circ u| |du|^2 |\tau(u)|^{p+q} &\geq 2(p-1) \int_M \varphi |\tau(u)|^{p+q-1} \langle \nabla \varphi, \nabla |\tau(u)| \rangle \\ &\quad + (p-1)(q+1) \int_M \varphi^2 |\tau(u)|^{p+q-2} |\nabla |\tau(u)||^2. \end{aligned}$$

Since

$$2 \int_M \varphi |\tau(u)|^{p+q-1} \langle \nabla \varphi, \nabla |\tau(u)| \rangle \geq -\delta \int_M \varphi^2 |\tau(u)|^{p+q-2} |\nabla |\tau(u)||^2 - \frac{1}{\delta} \int_M |\tau(u)|^{p+q} |\nabla \varphi|^2$$

for all $\delta > 0$, we have

$$\begin{aligned} [(p-1)(q+1) - \delta(p-1)] \int_M \varphi^2 |\tau(u)|^{p+q-2} |\nabla |\tau(u)||^2 &\leq \frac{p-1}{\delta} \int_M |\tau(u)|^{p+q} |\nabla \varphi|^2 \\ (2.3) \quad &\quad + \int_M \varphi^2 |R^N \circ u| |du|^2 |\tau(u)|^{p+q}. \end{aligned}$$

3 Rigidity results for p -biharmonic maps

In this section, we study p -biharmonic maps under the condition that the domain manifold (M^n, g) satisfies the following Sobolev inequality:

$$(3.1) \quad \left(\int_M \varphi^{\frac{2n}{n-2}} \right)^{\frac{n-2}{n}} \leq C_s \int_M |\nabla \varphi|^2 \text{ for all } \varphi \in C_0^1(M).$$

PROOF OF THEOREM 1.1. By the Hölder inequality and (3.1), we have

$$\begin{aligned} &\int_M \varphi^2 |R^N \circ u| |du|^2 |\tau(u)|^{p+q} \\ &\leq \left[\int_M (|R^N \circ u| \cdot |du|^2)^{\frac{n}{2}} \right]^{\frac{2}{n}} \left[\int_M (\varphi^2 |\tau(u)|^{p+q})^{\frac{n-2}{n}} \right]^{\frac{n-2}{n}} \\ &\leq C_s \| |R^N \circ u| \cdot |du|^2 \|_{L^{\frac{n}{2}}(M)} \int_M |\nabla (\varphi |\tau(u)|^{\frac{p+q}{2}})|^2 \\ &= C_s \| |R^N \circ u| \cdot |du|^2 \|_{L^{\frac{n}{2}}(M)} \left[\frac{(p+q)^2}{4} \int_M \varphi^2 |\tau(u)|^{p+q-2} |\nabla |\tau(u)||^2 + \int_M |\tau(u)|^{p+q} |\nabla \varphi|^2 \right] \\ &\quad + C_s \| |R^N \circ u| \cdot |du|^2 \|_{L^{\frac{n}{2}}(M)} (p+q) \int_M \varphi |\tau(u)|^{p+q-1} \langle \nabla |\tau(u)|, \nabla \varphi \rangle \\ &\leq C_s \| |R^N \circ u| \cdot |du|^2 \|_{L^{\frac{n}{2}}(M)} \left(\frac{(p+q)^2}{4} + \frac{\alpha(p+q)}{2} \right) \int_M \varphi^2 |\tau(u)|^{p+q-2} |\nabla |\tau(u)||^2 \\ &\quad + C_s \| |R^N \circ u| \cdot |du|^2 \|_{L^{\frac{n}{2}}(M)} \left(1 + \frac{p+q}{2\alpha} \right) \int_M |\tau(u)|^{p+q} |\nabla \varphi|^2 \end{aligned}$$

for all $\alpha > 0$. Here we used the Cauchy-Schwarz inequality in the last inequality. Combining the above inequality and (2.3), we obtain

$$(3.2) \quad A \int_M \varphi^2 |\tau(u)|^{Q-2} |\nabla |\tau(u)||^2 \leq B \int_M |\tau(u)|^Q |\nabla \varphi|^2,$$

where the constants A , B , and Q are defined by

$$A = (p-1)(Q+1-p) - \delta(p-1) - C_s \| |R^N \circ u| \cdot |du|^2 \|_{L^{\frac{n}{2}}(M)} \left(\frac{Q^2}{4} + \frac{\alpha Q}{2} \right),$$

$$B = \frac{p-1}{\delta} + C_s \| |R^N \circ u| \cdot |du|^2 \|_{L^{\frac{n}{2}}(M)} \left(1 + \frac{Q}{2\alpha} \right),$$

$$Q = p + q.$$

Moreover, since $\| |R^N \circ u| \cdot |du|^2 \|_{L^{\frac{n}{2}}(M)} < \frac{4(p-1)(Q+1-p)}{Q^2 C_s}$ by our assumption, we have

$$(p-1)(Q+1-p) - C_s \| |R^N \circ u| \cdot |du|^2 \|_{L^{\frac{n}{2}}(M)} \cdot \frac{Q^2}{4} > 0.$$

Thus, for δ and α small enough, we see that $A > 0$. Therefore (3.2) yields

$$\begin{aligned} A \int_{B_{x_0}(r)} |\tau(u)|^{Q-2} |\nabla |\tau(u)||^2 &\leq A \int_M \varphi^2 |\tau(u)|^{Q-2} |\nabla |\tau(u)||^2 \\ &\leq B \int_M |\tau(u)|^Q |\nabla \varphi|^2 \\ &\leq \frac{4B}{r^2} \int_{B_{x_0}(2r)} |\tau(u)|^Q. \end{aligned}$$

Letting r tend to ∞ , we see that $|\tau(u)|^{Q-2} |\nabla |\tau(u)||^2 = 0$ on M , which implies that $|\tau(u)| = \text{constant}$. Since the volume of M is infinite, we conclude that $\tau(u) = 0$. $\square$

By the proof of Theorem 1.1, when $L^{\frac{n}{2}}$ -norm of the sectional curvature of the image $u(M) \subset N$ is sufficiently small or L^n -norm of $|du|$ is sufficiently small, we can obtain two consequences as follows.

COROLLARY 3.1. *Let $u : (M^n, g) \rightarrow (N^m, h)$ be a p -biharmonic map from a complete noncompact Riemannian manifold (M^n, g) into a Riemannian manifold (N^m, h) with $|du| \leq C$ for some constant $C > 0$ and $\int_M |\tau(u)|^Q < \infty$ for some constant $Q > p - 1$. Assume that (M^n, g) satisfies the Sobolev inequality (3.1) and*

$$\| |R^N \circ u| \|_{L^{\frac{n}{2}}(M)} < \frac{4(p-1)(Q+1-p)}{Q^2 C^2 C_s},$$

where C_s denotes the Sobolev constant. Then u is harmonic.

COROLLARY 3.2. *Let $u : (M^n, g) \rightarrow (N^m, h)$ be a p -biharmonic map from a complete noncompact Riemannian manifold (M^n, g) into a Riemannian manifold (N^m, h) , with $|R^N \circ u| \leq D$ for some constant $D > 0$ and $\int_M |\tau(u)|^Q < \infty$ for $Q > p - 1$. Assume that (M^n, g) satisfies the Sobolev inequality (3.1) and*

$$\| |du| \|_{L^n(M)}^2 < \frac{4(p-1)(Q+1-p)}{Q^2 D C_s},$$

where C_s denotes the Sobolev constant. Then u is harmonic.

Now we prove Theorem 1.2.

PROOF OF THEOREM 1.2. By the Hölder inequality and (3.1), we have

$$\begin{aligned} &\int_M \varphi^2 |R^N \circ u| |du|^2 |\tau(u)|^{p+q} \\ &\leq \left[\int_M (|R^N \circ u|)^n \right]^{\frac{1}{n}} \left[\int_M (|du|^2)^n \right]^{\frac{1}{n}} \left[\int_M (\varphi^2 |\tau(u)|^{p+q})^{\frac{n}{n-2}} \right]^{\frac{n-2}{n}} \\ &\leq C_s \| |R^N \circ u| \|_{L^n(M)} \| |du|^2 \|_{L^n(M)} \int_M |\nabla \left(\varphi |\tau(u)|^{\frac{p+q}{2}} \right)|^2 \\ &= C_s \| |R^N \circ u| \|_{L^n(M)} \| |du|^2 \|_{L^n(M)} \left[\frac{(p+q)^2}{4} \int_M \varphi^2 |\tau(u)|^{p+q-2} |\nabla |\tau(u)||^2 + \int_M |\tau(u)|^{p+q} |\nabla \varphi|^2 \right] \end{aligned}$$

$$\begin{aligned}
& + C_s \|R^N \circ u\|_{L^n(M)} \| |du|^2 \|_{L^n(M)} \int_M \varphi |\tau(u)|^{p+q-1} \langle \nabla |\tau(u)|, \nabla \varphi \rangle \\
& \leq C_s \|R^N \circ u\|_{L^n(M)} \| |du|^2 \|_{L^n(M)} \left(\frac{(p+q)^2}{4} + \frac{\alpha(p+q)}{2} \right) \int_M \varphi^2 |\tau(u)|^{p+q-2} |\nabla |\tau(u)||^2 \\
& + C_s \|R^N \circ u\|_{L^n(M)} \| |du|^2 \|_{L^n(M)} \left(1 + \frac{p+q}{2\alpha} \right) \int_M |\tau(u)|^{p+q} |\nabla \varphi|^2
\end{aligned}$$

for all $\alpha > 0$. Therefore (2.3) yields

$$A \int_M \varphi^2 |\tau(u)|^{Q-2} |\nabla |\tau(u)||^2 \leq B \int_M |\tau(u)|^Q |\nabla \varphi|^2,$$

where the constants A , B , and Q are defined by

$$\begin{aligned}
A &= (p-1)(Q+1-p) - \delta(p-1) - C_s \|R^N \circ u\|_{L^n(M)} \cdot \| |du|^2 \|_{L^n(M)} \left(\frac{Q^2}{4} + \frac{\alpha Q}{2} \right), \\
B &= \frac{p-1}{\delta} + C_s \|R^N \circ u\|_{L^n(M)} \cdot \| |du|^2 \|_{L^n(M)} \left(1 + \frac{Q}{2\alpha} \right), \\
Q &= p+q.
\end{aligned}$$

In the same manner as in the proof of Theorem 1.1, we get the conclusion. $\square$

In particular, when $p = Q = 2$, we obtain the following harmonicity of the biharmonic.

COROLLARY 3.3. *Let $u : (M^n, g) \rightarrow (N^m, h)$ be a biharmonic map from a complete noncompact Riemannian manifold (M^n, g) into a Riemannian manifold (N^m, h) with $\int_M |\tau(u)|^2 < \infty$. Assume that (M^n, g) satisfies the Sobolev inequality (3.1) and*

$$\|R^N \circ u\|_{L^n(M)} \| |du|^2 \|_{L^n(M)} < \frac{1}{C_s},$$

where C_s denotes the Sobolev constant. Then u is harmonic.

Now we give a proof of Theorem 1.3.

PROOF OF THEOREM 1.3. By the assumption $|R^N \circ u| \cdot |du|^2 \leq K$, we have

$$\begin{aligned}
& \int_M \varphi^2 |R^N \circ u| \cdot |du|^2 |\tau(u)|^{p+q} \\
& \leq K \int_M \varphi^2 |\tau(u)|^{p+q} \\
& \leq \frac{K}{\lambda_1(M)} \int_M |\nabla \left(\varphi |\tau(u)|^{\frac{p+q}{2}} \right)|^2 \\
& \leq \frac{K}{\lambda_1(M)} \left[\frac{(p+q)^2}{4} \int_M \varphi^2 |\tau(u)|^{p+q-2} |\nabla |\tau(u)||^2 + \int_M |\tau(u)|^{p+q} |\nabla \varphi|^2 \right] \\
& + \frac{K}{\lambda_1(M)} (p+q) \int_M \varphi |\tau(u)|^{p+q-1} \langle \nabla |\tau(u)|, \nabla \varphi \rangle \\
& \leq \frac{K}{\lambda_1(M)} \left(\frac{(p+q)^2}{4} + \frac{\alpha(p+q)}{2} \right) \int_M \varphi^2 |\tau(u)|^{p+q-2} |\nabla |\tau(u)||^2 \\
& + \frac{K}{\lambda_1(M)} \left(1 + \frac{p+q}{2\alpha} \right) \int_M |\tau(u)|^{p+q} |\nabla \varphi|^2
\end{aligned}$$

for all $\alpha > 0$. Hence (2.3) yields

$$A \int_M \varphi^2 |\tau(u)|^{p+q-2} |\nabla |\tau(u)||^2 \leq B \int_M |\tau(u)|^{p+q} |\nabla \varphi|^2,$$

where the constants A and B are given by

$$A = (p-1)(q+1) - \delta(p-1) - \frac{K}{\lambda_1(M)} \left(\frac{(p+q)^2}{4} + \frac{\alpha(p+q)}{2} \right),$$

$$B = \frac{p-1}{\delta} + \frac{K}{\lambda_1(M)} \left(1 + \frac{p+q}{2\alpha} \right).$$

By repeating the arguments in Theorem 1.1, we get the conclusion. $\square$

Using Theorem 1.3, we have the following.

COROLLARY 3.4. *Let $u : (M^n, g) \rightarrow (N^m, h)$ be a p -biharmonic map from a complete noncompact Riemannian manifold (M^n, g) into a Riemannian manifold (N^m, h) with $|du| \leq C$, $|R^N \circ u| \leq D$ for some constants $C > 0, D > 0$ and $\int_M |\tau(u)|^Q < \infty$ for $Q > p-1$. Assume that the fundamental tone of M satisfies $\lambda_1(M) > \frac{Q^2 C^2 D}{4(p-1)(Q+1-p)}$. Then u is harmonic.*

On the other hand, if the Weyl curvature tensor W^N vanishes, then we have the following decomposition

$$R^N = \frac{S^N}{2m(m-1)} h \otimes h + \frac{1}{2} Z^N \otimes h,$$

where S^N denotes the scalar curvature and $Z^N = \text{Ric}^N - \frac{S^N}{m} h$ denotes the traceless Ricci tensor of (N^m, h) . Thus we have

$$\begin{aligned} & \sum_{i=1}^n \langle R^N(du(e_i), \tau(u)) du(e_i), \tau(u) \rangle \\ &= \frac{S^N}{m(m-1)} \left(|du|^2 |\tau(u)|^2 - \sum_{i=1}^n \langle du(e_i), \tau(u) \rangle_h \right) \\ &+ \frac{1}{m-2} \left(\sum_{i=1}^n Z^N(du(e_i), du(e_i) |\tau(u)|^2 + Z^N(\tau(u), \tau(u) |du|^2) \right) \\ &- \frac{2}{m-2} \left(\sum_{i=1}^n Z^N(du(e_i), \tau(u)) \langle du(e_i), \tau(u) \rangle_h \right) \\ &\leq \frac{S^N}{m(m-1)} \left(|du|^2 |\tau(u)|^2 - \sum_{i=1}^n \langle du(e_i), \tau(u) \rangle_h \right) + \frac{4}{m-2} |Z^N \circ u| |du|^2 |\tau(u)|^2. \end{aligned}$$

Hence, if we assume that N has nonpositive scalar curvature, i.e., $S^N \leq 0$, then we have

$$\sum_{i=1}^n \langle R^N(du(e_i), \tau(u)) du(e_i), \tau(u) \rangle \leq \frac{4}{m-2} |Z^N \circ u| |du|^2 |\tau(u)|^2.$$

This implies

$$\sum_{i=1}^n \langle R^N(du(e_i), |\tau(u)|^{p-2} \tau(u)) du(e_i), |\tau(u)|^{p-2} \tau(u) \rangle \leq \frac{4}{m-2} |Z^N \circ u| |du|^2 |\tau(u)|^{2(p-1)}.$$

Combining the above inequality and (2.1), we have

$$\frac{1}{2} \Delta |\tau(u)|^{2(p-1)} \geq |\nabla (|\tau(u)|^{p-2} \tau(u))|^2 - \frac{4}{m-2} |Z^N \circ u| |du|^2 |\tau(u)|^{2(p-1)}.$$

Applying the Kato inequality, the above inequality yields

$$|\tau(u)| \Delta |\tau(u)|^{p-1} \geq -\frac{4}{m-2} |Z^N \circ u| |du|^2 |\tau(u)|^p.$$

For the function φ in (2.2), we have

$$\int_M \varphi^2 |\tau(u)|^{q+1} \Delta |\tau(u)|^{p-1} \geq -\frac{4}{m-2} \int_M \varphi^2 |Z^N \circ u| |du|^2 |\tau(u)|^{p+q}.$$

Using the divergence theorem and Young's inequality, we get

$$(3.3) \quad \begin{aligned} & [(p-1)(q+1) - \delta(p-1)] \int_M \varphi^2 |\tau(u)|^{p+q-2} |\nabla |\tau(u)||^2 \\ & \leq \frac{p-1}{\delta} \int_M |\tau(u)|^{p+q} |\nabla \varphi|^2 + \frac{4}{m-2} \int_M \varphi^2 |Z^N \circ u| |du|^2 |\tau(u)|^{p+q} \end{aligned}$$

for any $\delta > 0$. Using (3.3), we are able to prove Theorem 1.4.

PROOF OF THEOREM 1.4. By the Hölder inequality and (3.1), we have

$$\begin{aligned} & \int_M \varphi^2 |Z^N \circ u| |du|^2 |\tau(u)|^{p+q} \\ & \leq C_s \| |Z^N \circ u| \cdot |du|^2 \|_{L^{\frac{n}{2}}(M)} \int_M |\nabla \left(\varphi |\tau(u)|^{\frac{p+q}{2}} \right)|^2 \\ & \leq C_s \| |Z^N \circ u| \cdot |du|^2 \|_{L^{\frac{n}{2}}(M)} \left(\frac{(p+q)^2}{4} + \frac{\alpha(p+q)}{2} \right) \int_M \varphi^2 |\tau(u)|^{p+q-2} |\nabla |\tau(u)||^2 \\ & \quad + C_s \| |Z^N \circ u| \cdot |du|^2 \|_{L^{\frac{n}{2}}(M)} \left(1 + \frac{p+q}{2\alpha} \right) \int_M |\tau(u)|^{p+q} |\nabla \varphi|^2 \end{aligned}$$

for all $\alpha > 0$. Combining the above inequality and (3.3), we obtain

$$A \int_M \varphi^2 |\tau(u)|^{Q-2} |\nabla |\tau(u)||^2 \leq B \int_M |\tau(u)|^Q |\nabla \varphi|^2,$$

where the constants A , B , and Q are defined by

$$\begin{aligned} A &= (p-1)(Q+1-p) - \delta(p-1) - \frac{4}{m-2} C_s \| |Z^N \circ u| \cdot |du|^2 \|_{L^{\frac{n}{2}}(M)} \left(\frac{Q^2}{4} + \frac{\alpha Q}{2} \right), \\ B &= \frac{p-1}{\delta} + \frac{4}{m-2} C_s \| |Z^N \circ u| \cdot |du|^2 \|_{L^{\frac{n}{2}}(M)} \left(1 + \frac{Q}{2\alpha} \right), \\ Q &= p+q. \end{aligned}$$

Using the same argument in the proof of Theorem 1.1, we get the conclusion. $\square$

On the other hand, it is known that the Weyl curvature tensor $W^N = 0$ when $m = 3$ or $m \geq 4$ and (N^m, h) is locally conformally flat. Thus Theorem 1.4 implies the following.

COROLLARY 3.5. *Let $u : (M^n, g) \rightarrow (N^m, h)$ be a p -biharmonic map from a complete noncompact Riemannian manifold (M^n, g) into a Riemannian manifold (N^m, h) with $S^N \leq 0$ and $\int_M |\tau(u)|^Q < \infty$ for some constant $Q > p-1$. Assume that (M^n, g) satisfies the Sobolev inequality (3.1) and assume that either*

- (i) $m = 3$ or
- (ii) $m \geq 4$ and N is locally conformally flat.

If

$$\| |Z^N \circ u| \cdot |du|^2 \|_{L^{\frac{n}{2}}(M)} < \frac{(m-2)(p-1)(Q+1-p)}{Q^2 C_s},$$

where C_s denotes the Sobolev constant. Then u is harmonic.

By the proof of Theorem 1.4, it is easy to prove two following corollaries.

COROLLARY 3.6. *Let $u : (M^n, g) \rightarrow (N^m, h)$ be a p -biharmonic map from a complete noncompact Riemannian manifold (M^n, g) into a Riemannian manifold (N^m, h) with $W^N = 0, S^N \leq 0, |du| \leq C$ for some constant $C > 0$ and $\int_M |\tau(u)|^Q < \infty$ for $Q > p - 1$. Assume that (M^n, g) satisfies the Sobolev inequality (3.1) and*

$$\|Z^N \circ u\|_{L^{\frac{Q}{2}}(M)} < \frac{(m-2)(p-1)(Q+1-p)}{Q^2 C^2 C_s},$$

where C_s denotes the Sobolev constant. Then u is harmonic.

COROLLARY 3.7. *Let $u : (M^n, g) \rightarrow (N^m, h)$ be a p -biharmonic map from a complete noncompact Riemannian manifold (M^n, g) into a Riemannian manifold (N^m, h) with $W^N = 0, S^N \leq 0, |Z^N \circ u| \leq D$ for some constant $D > 0$ and $\int_M |\tau(u)|^Q < \infty$ for $Q > p - 1$. Assume that (M^n, g) satisfies the Sobolev inequality (3.1) and*

$$\|du\|_{L^n(M)}^2 < \frac{(m-2)(p-1)(Q+1-p)}{Q^2 D C_s},$$

where C_s denotes the Sobolev constant. Then u is harmonic.

Using the general Hölder inequality for three functions in the terms of curvature of (3.3) as in the proof of Theorem 1.2, we have the following result.

THEOREM 3.8. *Let $u : (M^n, g) \rightarrow (N^m, h)$ be a p -biharmonic map from a complete noncompact Riemannian manifold (M^n, g) into a Riemannian manifold (N^m, h) with $W^N = 0, S^N \leq 0$ and $\int_M |\tau(u)|^Q < \infty$ for $Q > p - 1$. Assume that (M^n, g) satisfies the Sobolev inequality (3.1) and*

$$\|Z^N \circ u\|_{L^n(M)} \| |du|^2 \|_{L^n(M)} < \frac{(m-2)(p-1)(Q+1-p)}{Q^2 C_s},$$

where C_s denotes the Sobolev constant. Then u is harmonic.

In particular, if $p = Q = 2$, then we immediately obtain the following.

COROLLARY 3.9. *Let $u : (M^n, g) \rightarrow (N^m, h)$ be a biharmonic map from a complete noncompact Riemannian manifold (M^n, g) into a Riemannian manifold (N^m, h) , with $W^N = 0, S^N \leq 0$ and $\int_M |\tau(u)|^2 < \infty$. Assume that (M^n, g) satisfies the Sobolev inequality (3.1) and*

$$\|Z^N \circ u\|_{L^n(M)} \| |du|^2 \|_{L^n(M)} < \frac{m-2}{4C_s},$$

where C_s denotes the Sobolev constant. Then u is harmonic.

PROOF OF THEOREM 1.6. By using (3.3) and repeating the arguments in Theorem 1.3, we complete the proof of Theorem 1.6. $\square$

COROLLARY 3.10. *Let $u : (M^n, g) \rightarrow (N^m, h)$ be a p -biharmonic map from a complete noncompact Riemannian manifold (M^n, g) into a Riemannian manifold (N^m, h) with $W^N = 0, S^N \leq 0, |du| \leq C, |Z^N \circ u| \leq D$ for some constants $C > 0, D > 0$ and $\int_M |\tau(u)|^Q < \infty$ for $Q > p - 1$. Assume that $\lambda_1(M) > \frac{C^2 D Q^2}{(m-2)(p-1)(Q+1-p)}$. Then u is harmonic.*

4 Rigidity results for p -biharmonic submanifolds

We recall that if $u : (M^n, g) \rightarrow (N^m, h)$ is an isometric immersion, then u is called a p -biharmonic submanifold. In particular, a 2-biharmonic submanifold is called a *biharmonic* submanifold. The second fundamental form $B : TM \times TM \rightarrow T^\perp M$ is defined by:

$$\bar{\nabla}_X Y = \nabla_X Y + B(X, Y)$$

for $X, Y \in \Gamma(TM)$, where $\bar{\nabla}$ is the Levi-Civita connection on N and ∇ is the Levi-Civita connection on M . The Weingarten formula is given by

$$\bar{\nabla}_X \xi = -A_\xi X + \nabla_X^\perp \xi$$

for $X \in \Gamma(TM)$, where A_ξ is called the Weingarten map with respect to $\xi \in T^\perp M$ and $\nabla^\perp$ denotes the normal connection on the normal bundle of M in N . For any $x \in M$, the mean curvature vector field H of M at x is

$$\vec{H} = \frac{1}{n} \sum_{i=1}^n B(e_i, e_i).$$

Now suppose $u : (M^n, g) \rightarrow (N^m, h)$ is a p -biharmonic isometric immersion of a Riemannian manifold M with mean curvature vector $\vec{H}$ into a Riemannian manifold N . Then we have

$$\tau(u) = n\vec{H}.$$

Moreover, the p -biharmonic submanifold u satisfies the following equation:

$$\tau_p(u) := \Delta(|\vec{H}|^{p-2}\vec{H}) - \sum_{i=1}^n R^N(|\vec{H}|^{p-2}\vec{H}, e_i)e_i = 0.$$

Now we are ready to prove Theorem 1.7.

PROOF OF THEOREM 1.7. Let B be the second fundamental form of M and A be the Weingarten map. Then, by the Bochner-Weitzenböck formula, we have (see [CL16, Han15] for example)

$$\begin{aligned} \Delta|\vec{H}|^{2p-2} &= 2|\nabla(|\vec{H}|^{p-2}\vec{H})|^2 + 2\langle \Delta(|\vec{H}|^{p-2}\vec{H}), |\vec{H}|^{p-2}\vec{H} \rangle \\ &= 2|\nabla(|\vec{H}|^{p-2}\vec{H})|^2 + 2\left\langle \sum_{i=1}^n B(A_{|\vec{H}|^{p-2}\vec{H}} e_i, e_i), |\vec{H}|^{p-2}\vec{H} \right\rangle \\ &\quad + 2\sum_{i=1}^n \left\langle R^N(e_i, |\vec{H}|^{p-2}\vec{H})e_i, |\vec{H}|^{p-2}\vec{H} \right\rangle. \end{aligned} \tag{4.1}$$

On the other hand, we have (see [Han15] for instance)

$$\left\langle \sum_{i=1}^n B(A_{|\vec{H}|^{p-2}\vec{H}} e_i, e_i), |\vec{H}|^{p-2}\vec{H} \right\rangle \geq n|\vec{H}|^{2p}, \tag{4.2}$$

and

$$\sum_{i=1}^n \left\langle R^N(e_i, |\vec{H}|^{p-2}\vec{H})e_i, |\vec{H}|^{p-2}\vec{H} \right\rangle \geq -|\vec{H}|^{2p-2}|R^N \circ u|. \tag{4.3}$$

Combining (4.1)-(4.3), we obtain

$$\Delta|\vec{H}|^{2p-2} \geq 2|\nabla(|\vec{H}|^{p-2}\vec{H})|^2 + 2n|\vec{H}|^{2p} - 2|\vec{H}|^{2p-2}|R^N \circ u|.$$

By the Kato inequality, we have

$$|\vec{H}|\Delta|\vec{H}|^{p-1} \geq n|\vec{H}|^{p+2} - |\vec{H}|^p |R^N \circ u|.$$

For the function φ in (2.2), we have

$$\int_M \varphi^2 |\vec{H}|^{q+1} \Delta |\vec{H}|^{p-1} \geq n \int_M \varphi^2 |\vec{H}|^{p+q+2} - \int_M \varphi^2 |\vec{H}|^{p+q} |R^N \circ u|.$$

Using integration by parts, the above inequality implies

$$(4.4) \quad \begin{aligned} \int_M \varphi^2 |\vec{H}|^{p+q} |R^N \circ u| &\geq [(p-1)(q+1) - \delta(p-1)] \int_M \varphi^2 |\vec{H}|^{p+q-2} |\nabla |\vec{H}||^2 \\ &\quad + n \int_M \varphi^2 |\vec{H}|^{p+q+2} - \frac{p-1}{\delta} \int_M |\vec{H}|^{p+q} |\nabla \varphi|^2 \end{aligned}$$

for all $\delta > 0$. Here we used the Cauchy-Schwarz inequality in (4.4). By the Hölder inequality and the Sobolev inequality (3.1), we get

$$\begin{aligned} \int_M \varphi^2 |\vec{H}|^{p+q} |R^N \circ u| &\leq C_s \|R^N \circ u\|_{L^{\frac{n}{2}}(M)} \int_M |\nabla(\varphi |\vec{H}|^{\frac{p+q}{2}})|^2 \\ &\leq C_s \|R^N \circ u\|_{L^{\frac{n}{2}}(M)} \left(\frac{(p+q)^2}{4} + \frac{\alpha(p+q)}{2} \right) \int_M \varphi^2 |\vec{H}|^{p+q+2} \\ &\quad + C_s \|R^N \circ u\|_{L^{\frac{n}{2}}(M)} \left(1 + \frac{p+q}{2\alpha} \right) \int_M |\vec{H}|^{p+q} |\nabla \varphi|^2. \end{aligned}$$

Therefore (4.4) implies

$$C \int_M \varphi^2 |\vec{H}|^{Q-2} |\nabla |\vec{H}||^2 + n \int_M \varphi^2 |\vec{H}|^{Q+2} \leq D \int_M |\vec{H}|^Q |\nabla \varphi|^2$$

where the constants C , D , and Q are defined by

$$\begin{aligned} C &= (p-1)(Q+1-p) - \delta(p-1) - C_s \|R^N \circ u\|_{L^{\frac{n}{2}}(M)} \left(\frac{Q^2}{4} + \frac{\alpha Q}{2} \right), \\ D &= \frac{p-1}{\delta} + C_s \|R^N \circ u\|_{L^{\frac{n}{2}}(M)} \left(1 + \frac{Q}{2\alpha} \right), \\ Q &= p+q. \end{aligned}$$

As before, we can conclude that $|\vec{H}| = 0$. □

PROOF OF THEOREM 1.8. By using (4.4) and applying the same argument as in the proof of Theorem 1.3, we are able to prove Theorem 1.8. □

Finally we prove Theorem 1.9.

PROOF OF THEOREM 1.9. Let ν be the unit normal vector field of M . Then we have

$$\tau(u) = n\vec{H} = n|\vec{H}|\nu.$$

Since (N^{n+1}, h) is Einstein, we have

$$\begin{aligned} \sum_{i=1}^n \langle R^N(e_i, |\vec{H}|^{p-2} \vec{H}) e_i, |\vec{H}|^{p-2} \vec{H} \rangle &= \sum_{i=1}^n |\vec{H}|^{2p-2} \langle R^N(e_i, \nu) e_i, \nu \rangle \\ &= -|\vec{H}|^{2p-2} \text{Ric}^N(\nu, \nu) \end{aligned}$$

$$= -\frac{S^N}{n+1}|\vec{H}|^{2p-2}.$$

Combining inequalities (4.1), (4.2) and the above equality, we obtain

$$\Delta|\vec{H}|^{2p-2} \geq 2|\nabla(|\vec{H}|^{p-2}\vec{H})|^2 + 2n|\vec{H}|^{2p} - 2\frac{S^N}{n+1}|\vec{H}|^{2p-2}.$$

By the Kato inequality, we have

$$|\vec{H}|\Delta|\vec{H}|^{p-1} \geq n|\vec{H}|^{p+2} - \frac{S^N}{n+1}|\vec{H}|^p.$$

For the function φ in (2.2), we have

$$\int_M \varphi^2 |\vec{H}|^{q+1} \Delta|\vec{H}|^{p-1} \geq n \int_M \varphi^2 |\vec{H}|^{p+q+2} - \frac{S^N}{n+1} \int_M \varphi^2 |\vec{H}|^{p+q}.$$

Using integration by parts and the Cauchy-Schwarz inequality, the above inequality implies

$$(4.5) \quad \begin{aligned} \frac{S^N}{n+1} \int_M \varphi^2 |\vec{H}|^{p+q} &\geq [(p-1)(q+1) - \delta(p-1)] \int_M \varphi^2 |\vec{H}|^{p+q-2} |\nabla|\vec{H}||^2 \\ &\quad + n \int_M \varphi^2 |\vec{H}|^{p+q+2} - \frac{p-1}{\delta} \int_M |\vec{H}|^{p+q} |\nabla\varphi|^2 \end{aligned}$$

for all $\delta > 0$. Moreover, we have

$$\begin{aligned} \frac{S^N}{n+1} \int_M \varphi^2 |\vec{H}|^{p+q} &\leq \frac{S^N}{(n+1)\lambda_1(M)} \int_M |\nabla(\varphi|\vec{H}|^{\frac{p+q}{2}})|^2 \\ &\leq \frac{S^N}{(n+1)\lambda_1(M)} \left(\frac{(p+q)^2}{4} + \frac{\alpha(p+q)}{2} \right) \int_M \varphi^2 |\vec{H}|^{p+q+2} \\ &\quad + \frac{S^N}{(n+1)\lambda_1(M)} \left(1 + \frac{p+q}{2\alpha} \right) \int_M |\vec{H}|^{p+q} |\nabla\varphi|^2. \end{aligned}$$

Therefore (4.5) implies

$$C \int_M \varphi^2 |\vec{H}|^{Q-2} |\nabla|\vec{H}||^2 + n \int_M \varphi^2 |\vec{H}|^{Q+2} \leq D \int_M |\vec{H}|^Q |\nabla\varphi|^2$$

where the constants C , D , and Q are defined by

$$\begin{aligned} C &= (p-1)(Q+1-p) - \delta(p-1) - \frac{S^N}{(n+1)\lambda_1(M)} \left(\frac{Q^2}{4} + \frac{\alpha Q}{2} \right), \\ D &= \frac{p-1}{\delta} + \frac{S^N}{(n+1)\lambda_1(M)} \left(1 + \frac{Q}{2\alpha} \right), \\ Q &= p+q. \end{aligned}$$

Using the same argument as in the proof of Theorem 1.1, we can obtain that $|\vec{H}| = 0$, which implies that u is minimal. $\square$

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William Barker

DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF ARKANSAS AT LITTLE ROCK,
LITTLE ROCK, 72204, USA

E-mail address: wkbarker@ualr.edu

Nguyen Thac Dung

FACULTY OF MATHEMATICS - MECHANICS - INFORMATICS
VIETNAM NATIONAL UNIVERSITY, UNIVERSITY OF SCIENCE, HANOI
HANOI, VIETNAM.

E-mail address: dungmath@vnu.edu.vn or dungmath@gmail.com

Keomkyo Seo

DEPARTMENT OF MATHEMATICS AND RESEARCH INSTITUTE OF NATURAL SCIENCES
SOOKMYUNG WOMEN'S UNIVERSITY
CHEONGPA-RO 47-GIL 100, YONGSAN-KU, SEOUL, 04310, KOREA

E-mail address: kseo@sookmyung.ac.kr

Nguyen Dang Tuyen

DEPARTMENT OF MATHEMATICS
HANOI UNIVERSITY OF CIVIL ENGINEERING
HANOI, VIETNAM

E-mail address: tuyennd@huce.edu.vn