NEW CHARACTERIZATIONS OF QUASI-FROBENIUS RINGS

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Abstract. In this paper, we firstly provide several new characterizations of quasi-Frobenius rings by using some generalized injectivity of rings with certain chain conditions. Namely, we prove among other results, that: (1) A ring R is quasi-Frobenius if and only if R is right C_{11}, right minfull with ACC on right annihilators; (2) A ring R is quasi-Frobenius if and only if R is two-sided min-CS with ACC on right annihilators in which $Soc(_RR) \leq_e R_R$; (3) A ring R is quasi-Frobenius if and only if R is right Johns left C_{11}; (4) A ring R is quasi-Frobenius if and only if R is quasi-dual two-sided C_{11} with ACC on right annihilators. Moroever, we give more characterizations of quasi-Frobenius rings. For example, it is shown that a ring R is quasi-Frobenius if and only if R is a left P-injective left IN-ring with right RMC and $Z(R_R) = Z(_RR)$. Also, we prove that if R is a right duo, right QF-3^+ left quasi-duo ring satisfying ACC on right annihilators, then R is quasi-Frobenius. In this paper, several known results on quasi-Frobenius rings are reproved as corollaries.

1. Introduction

Throughout this paper, all rings R are associative with identity and all modules are unitary right R-module. The notations $N \leq_e M$ and $N \leq^{\oplus} M$ mean that N is an essential submodule and a direct summand, respectively. Let M be an R-module. Recall that the *singular submodule* $Z(M)$ of M is defined by

$$Z(M) = \{m \in M \mid mI = 0 \text{ for some essential right ideal } I \text{ of } R\}.$$

The *Goldie torsion submodule* $Z_2(M)$ of M (also known as the *second singular submodule* of M) is defined to be the submodule of M which contains $Z(M)$ such that $Z(M/Z(M)) = Z_2(M)/Z(M)$. The module M is called *singular* if $Z(M) = M$ and is called *nonsingular* if $Z(M) = 0$ (equivalently, $Z_2(M) = 0$). Recall that $M/Z_2(M)$ is a nonsingular module. For a ring R, we denote by $J(R)$ the Jacobson radical of R. If X is a subset of a ring R, the right (left) annihilator in R is denoted by $r(X)$ $(l(X))$.

The notion of self-injective rings is generalized by many authors. In [12], let R be a ring, then

• R is called right P-injective (resp., 2-injective) ring if every R-homomorphism from a principal (resp., 2-generated) right ideal of R extends to an endomorphims of R.

• R is said to be right mininjective if every R-homomorphism from a minimal right ideal of R extends to an endomorphims of R.

Key words and phrases. automorphism-invariant ring, C_{11}-rings, mininjective ring, IN-ring, P-injective ring, quasi-Frobenius ring.

• R is called right simple-injective if every R-linear map with simple image from a right ideal to R extends to R.

• R is called right dual if $rl(T) = T$ for all right ideals T of R.

• R is called right minfull if it is semiperfect right minijective and $Soc(eR) \neq 0$ for each local idempotent e of R.

• R is called right min-CS if every minimal right ideal is essential in a direct summand.

• R is said to be left IN ring if $r(T \cap T') = r(T) + r(T')$ for all left ideals T and T' of R .

A ring R is called right GP-injective if for each $0 \neq a \in R$, there exists $n \in \mathbb{N}$ such that $a^n \neq 0$ and $lr(a^n) = Ra^n$ ([2]).

Recall that a module M is said to be a C_{11}-module if every submodule of M has a complement which is a direct summand ([21]). A ring R is called a right C_{11}-ring if R_R is a C_{11}-module. Clearly, every CS-module satisfies the C_{11}-condition. However, the converse is not true in general (see [21, p. 1814]).

A submodule N of a module M is said to be an automorphism-invariant submodule if $f(N) \subseteq N$ for every automorphism f of M. A module is called automorphism-invariant if it is an automorphism-invariant of its injective hull ([14]). A ring R is called right automorphism-invariant if R_R is automorphism-invariant.

A module M is said to be satisfy the restricted minimum condition (briefly, RMC) if for every essential submodule N of M, M/N is an artinian module. A ring R is said to be have right RMC if R satisfies the RMC as a right R-module.

Recall that a ring R is quasi-Frobenius if R is two-sided artinian and two-sided self-injective. Quasi-Frobenius rings play an important role in the theory, and many interesting characterizations can be found in ([12]).

In Section 2, we provide several new characterizations of quasi-Frobenius rings by using some generalized injectivity of rings satisiying certain chain conditions. We first prove that a right C_{11}, right minfull ring satisfying ACC on right annihilators is quasi-Frobenius. We prove that a two-sided min-CS ring with ACC on right annihilators in which $Soc(_RR) \leq_e R_R$ is quasi-Frobenius. It is also shown that a left AGP-injective two-side min-CS ring satisfying ACC on left annihilators is quasi-Frobenius We prove that a right Johns left C_{11}-ring is quasi-Frobenius. Note that in this section, some known results on quasi-Frobenius are obtained as corollaries.

In section 3, quasi-Frobenius rings are characterized via two-side C_{11}-rings. We prove that a ring is quasi-Frobenius if and only if it is quasi-dual two-side C_{11} with ACC on right annihilators. Moroever, it is shown that a right artinian two-side C_{11}-ring R in which $Soc(R_R) = Soc(_RR)$ is quasi-Frobenius.

Section 4 is devoted to automorphism-invariant rings and their generalizations. In this section, it is shwon among others results that every left automorphism-invariant ring R with ACC on right annihilators in which $Soc(_RR)$ is an essential right ideal is quasi-Frobenius. We prove also that every two-side pseudo-c^*-injective two-side C_{11}-ring with ACC on right annihilators is quasi-Frobenius.

In section 5, we provide more characterizations of quasi-Frobenius rings. Firstly, we prove that a left perfect right simple-injective ring such that for every injective right R-module M, $Z_2(M)$ is projective, is quasi-Frobenius. Also, it is shown that a two-sided minfull left (or right) pseudo-coherent ring R for which $J(R)$ is left or right T-nilpotent is quasi-Frobenius. Moroever, we prove that a left P-injective left

IN-ring with right RMC is quasi-Frobenius if and only if $Z(_RR) = Z(R_R)$. This result extends Theorem 13((1) $\Leftrightarrow$ (2)) in [10] and Proposition 18.6 in [5]. As a direct consequence of the last result, it is shown that a two-sided P-injective left IN-ring with right RMC is quasi-Frobenius. Finally, we show that if R is a right duo, right QF-3^+ left quasi-duo ring satisfying ACC on right annihilators, then R is quasi-Frobenius.

2. Quasi-Frobenius rings via the mimimal ideals

It is obvious that a quasi-Frobenius ring is right minfull with ACC on right annihiltors. However Examples 2.5 and 6.41(1) in [12] show that the converse is not true in general. In the next theorem, we provide some conditions which force a right minfull ring with ACC on right annihiltors to be quasi-Frobenius. We first prove the following lemma.

Lemma 2.1. *Let R be a right C_{11} right minifull ring. Then $Soc(eR)$ is a minimal right ideal for every local idempotent e of R and R is right finitely cogenerated.*

Proof. Since R is right minfull, R_R satisfies the C_2-condition by [12, Lemma 1.46 and Theorem 3.12]. Now, let e be a local idempotent of R. As R_R is a C_{11}-module, then by [21, Theorem 4.3], eR is also a C_{11}-module. Hence, since eR is indecomposable, it follows from [21, Proposition 2.3(iii)] that eR is uniform. Note that $Soc(eR) \neq 0$. Therefore, $Soc(eR)$ is a minimal right ideal. On the other hand, since R is semiperfect, there exits a decomposition $R_R = e_1R \oplus e_2R \oplus .... \oplus e_nR$ where each e_i is a local idempotent. Therefore, by what we shown above, $Soc(e_iR)$ is a minimal right ideal and $Soc(e_iR) \leq_e e_iR$. From this, we deduce that $Soc(R_R)$ is a finitely generated right ideal and $Soc(R_R) \leq_e R_R$. Therefore, R is right finitely cogenerated. □

Theorem 2.2. *Then following conditions are equivalent for a ring R:*

1. *R is quasi-Frobenius;*
2. *R is right minfull with ACC on right annihilators and every complement right ideal is a right annihilator;*
3. *R is right C_{11} right minfull with ACC on right annihilators;*
4. *R is right C_{11} right minfull with right RMC.*

Proof. $(1) \Rightarrow (2), (4)$ are clear.

$(2) \Rightarrow (3)$ Being right minfull, R is left Kasch by [12, Theorem 3.12]. But every complement right ideal is a right annihilator. Then R is a right C_{11}-ring by [24, Theorem 10].

$(3) \Rightarrow (1)$ By Lemma 2.1, R is right finitely cogenerated. In addition, since R is right mininjective, $Soc(R_R) \subseteq Soc(_RR)$ by [12, Theorem 2.21]. Consequently, $Soc(_RR) \leq_e R_R$, and so $J(R) \subseteq Z(R)$. But R is semiperfect. Then $J(R) = Z(R)$ Note that R has ACC on right annihilators. Therefore, in view of [12, Lemma 3.29], $J(R)$ is nilpotent, from which it follows that R is semiprimary. Hence, by Lemma 2.1 and [22, Corollary 7], $Soc(Re)$ is a minimal left ideal for every local idempotent e of R. In addition, since R is right minfull, we infer from [12, Theorem 3.12] that R is right Kasch. So, using [12, Theorem 3.7(3)(a)], we deduce that $Soc(R_R) = Soc(_RR)$. Now, we claim that R is left mininjective. To see this fact, let e be a local idempotent of R. By Lemma 2.1, $Soc(eR)$ is a minimal right ideal. Therefore, being semiperfect,

R is left mininjective by [12, Theorem 3.2(1)]. Finally, since R is a right mininjective ring with ACC on right annihilators in which $Soc(R_R) \leq_e R_R$, R is quasi-Frobenius by [12, Theorem 3.31].

(4) $\Rightarrow$ (1) By Lemma 2.1, R is right finitely cogenerated. Thus, by hypothesis, $R/Soc(R_R)$ is right noetherian, and so R has ACC on right annihilators. Therefore, R is quasi-Frobenius by (3). $\square$

Corollary 2.3. *The following conditions are equivalent for a ring R:*

1. *R is quasi-Frobenius;*
2. *R is a right minfull right C_{11}-ring and $Z(R_R)$ is a noetherian right R-module.*

Proof. (1) $\Rightarrow$ (2) is clear.

(2) $\Rightarrow$ (1) Assume that R has the stated condition. Then by Lemma 2.1, $Soc(R_R)$ is a finitely generated right ideal and essential in R_R. So, using [12, Lemma 6.43], we deduce that $R/Z(R_R)$ is right noetherian. Note that $Z(R_R)$ is a noetherian right R-module. Hence, R is right noetherian, which implies that R has ACC on right annihilators. Therefore, according to Theorem 2.2(2), R is quasi-Frobenius. $\square$

Recall a ring R is called right (left) QF-2 if R is a direct sum of unform right (left) ideals.

Corollary 2.4 ([18, Theorem 4.4])**.** *If R is a QF-2 ring with ACC on right annihilators in which $Soc(_RR) \leq_e R_R$, then R is quasi-Frobenius.*

Proof. By [18, Lemma 4.3], R is semiperfect and $Soc(Re) \neq 0$ for every local idempotent $e \in R$. Since R is left QF-2, Re is uniform, from which it follows that $Soc(Re)$ is simple. In addition, since $Soc(_RR) \leq_e R_R$, $Soc(R_R) \subseteq Soc(_RR)$. So, R is right mininjective by [12, Proposition 3.5] and consequently, R is right minfull. Note that R is a right C_{11}-ring (being right QF-2) by [21, Theorem 2.4]. Therefore, the result follows from Theorem 2.2(2). $\square$

Corollary 2.5. *Then following conditions are equivalent for a ring R:*

1. *R is quasi-Frobenius;*
2. *R is right C_{11} right GP-injective with ACC on right annihilators;*
3. *R is a right artinian right mininjective right CS-ring;*
4. *R is a right artinian right mininjective right C_{11}-ring.*

Proof. (1) $\Rightarrow$ (2) is clear.

(2) $\Rightarrow$ (1) follows from [2, Theorem 3.7] and Theorem 2.2(2).

(1) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4) follows from Theorem 2.2(2). $\square$

Corollary 2.6. *The following conditions are equivalent for a ring R:*

1. *R is quasi-Frobenius;*
2. *R is right C_{11}, left minannihilator and right artinian.*

Theorem 2.7. *Then following conditions are equivalent for a ring R:*

1. *R is quasi-Frobenius;*
2. *R is two-sided min-CS with ACC on right annihilators in which $Soc(_RR)$ is essential in R_R.*
3. *R is left AGP-injective two-sided min-CS with ACC on left annihilators;*

Proof. (1) $\Rightarrow$ (2), (3) are clear.

(2) $\Rightarrow$ (1) Since R has ACC on right annihilators and $Soc(_RR) \leq_e R_R$, R is semiprimary by [18, Lemma 4.3]. Thus, R is left Kasch by [12, Lemma 4.2]. As R is left min-CS, then it follows from [12, Lemma 4.5] that $Soc(Re)$ is simple for all local idempotent $e \in R$. On the other hand, the fact that $Soc(_RR) \leq_e R_R$ implies that $Soc(R_R) \subseteq Soc(_RR)$. Hence, being semiperfect, R is right mininjective by [12, Proposition 3.5], from which it follows that R is right minfull. Thus, using [12, Theorem 3.12], R is right Kasch. Since R is semiperfect right min-CS, we infer from [12, Lemma 4.5] that $Soc(eR)$ is simple for all local idempotent $e \in R$ for. But we have already seen that $Soc(Re)$ is simple for all local idempotent $e \in R$. Then, since R is right Kasch, it follows from [12, Theorem 3.7(3)] that $Soc(R_R) = Soc(_RR)$. So, by [12, Proposition 3.5] again, R is left mininjective. Finally, being a two-sided mininjective ring with ACC on right annihilators in which $Soc(_RR) \leq_e R_R$, R is quasi-Frobenius by [12, Theorem 3.31].

(3) $\Rightarrow$ (1) Being left AGP-injective with ACC on left annihilators, R is semiprimary by [25, Corollary 1.6]. On the other hand, since R is left AGP-injective, $J(_RR) = Z(_RR)$ by [25, Lemma 1.3], and so $Soc(_RR) \subseteq Soc(R_R)$. This implies that $Soc(R_R) \leq_e {_RR}$. Therefore, according to (2) $\Rightarrow$ (1), R is quasi-Frobenius. $\square$

A module M is called ef-extending if every closed submodule which contains essentially a finitely generated submodule is a direct summand of M.

Corollary 2.8 ([18, Theorem 4.7])**.** *Then following conditions are equivalent for a ring R:*

1. *R is quasi-Frobenius;*
2. *R is right ef-extending with ACC on right annihilators in which $Soc(_RR) \leq_e R_R$.*

Proposition 2.9. *Then following conditions are equivalent for a ring R:*

1. *R is quasi-Frobenius;*
2. *R is a right noetherian left AGP-injective two-sided ef-extending ring.*

Proof. (1) $\Rightarrow$ (2) is clear.

(2) $\Rightarrow$ (1) Since R is right noetherian, R contains no infinite orthogonal sets of idempotents. Hence, $_RR = Re_i \oplus ... \oplus Re_n$, where each Re_i is indecomposable. As, $_RR$ is an ef-extending module, each Re_i is uniform. Thus, $_RR$ has finite uniform dimension. So, using [14, Corollary 1.2], we deduce that R is semilocal. On the other hand, being right noetherian left AGP-injective, $J(R)$ is nilpotent by [14, Theorem 2.1]. Therefore, R is semiprimary, from which it follows that R is right artinian. So, R has ACC on left annihilators. Therefore, the claim follows from Theorem 2.7(3). $\square$

Theorem 2.10. *Then following conditions are equivalent for a ring R:*

1. *R is quasi-Frobenius;*
2. *R is left C_{11} right cogenerator with ACC on right annihilators.*

Proof. (1) $\Rightarrow$ (2) is clear.

(2) $\Rightarrow$ (1) As R has ACC on right annihilators, then it has enough idempotents. So we can write $R = Re_1 \oplus Re_2 \oplus ... \oplus Re_n$, where $\{e_i\}_{i=1}^n$ is an orthogonal set of

primitive idempotents. Since R is right cogenerator, R is right Kasch. Thus, R is a left C_2-ring, and so $_RR$ is a C_3-module. Then, since $_RR$ is a C_{11}-module, it follows from [21, Proposition 2.3 (iii) and Theorem 4.3] that each Re_i is uniform. Consequently, $_RR$ has finite uniform dimension. As $_RR$ is a C_2-module, then R is semiperfect by [12, Lemma 4.26]. In particular, R has a finite number of isomorphism classes of simple right and (left) R-modules. Since R is right cogenerator, R is right self-injective by [12, Theorem 1.56]. Therefore, in view of [5, Proposition 18.9], R is quasi-Frobenius. □

Theorem 2.11. *Then following conditions are equivalent for a ring R:*

1. *R is quasi-Frobenius;*
2. *R is a right noetherian left P-injective lef C_{11}-ring.*

Proof. (1) ⇒ (2) is clear.

(2) ⇒ (1) Since R is right noetherian, R contains no infinite orthogonal sets of idempotents. So, we can write $_RR = Re_1 \oplus ... \oplus Re_n$, where each Re_i is a primitive orthogonal idempotent. Note that $_RR$ is a C_3-module. Then, since $_RR$ is a C_{11}-module, it follows from [21, Proposition 2.3(iii) and Theorem 4.3] that each Re_i is uniform. Consequently, $_RR$ has finite uniform dimension. Thus, using [25, Corollary 1.2], we deduce that R is semilocal. On the other hand, since R is right noetherian left AGP-injective, $J(R)$ is nilpotent by [25, Theorem 2.1]. This implies that R is semiprimary, and so R is right artinian. Hence, R has ACC on left annihilators. Note that R is left mininjective. Then, R is left minfull. Therefore, being left C_{11}, R is quasi-Frobenius by Theorem 2.2 (2). □

Corollary 2.12. *Then following conditions are equivalent for a ring R:*

1. *R is quasi-Frobenius;*
2. *R is a right Johns left C_{11}-ring.*

Corollary 2.13. *Then following conditions are equivalent for a ring R:*

1. *R is quasi-Frobenius;*
2. *R is a strongly right Johns left C_{11}-ring.*

3. Quasi-Frobenius rings via two-sided C_{11}-rings

Following [25], a ring R is called right (left) quasi-dual if every right (left) ideal is a direct summand of a right (left) annihilator.

Theorem 3.1. *Then following conditions are equivalent for a ring R:*

1. *R is quasi-Frobenius;*
2. *R is quasi-dual two-sided C_{11} with ACC on right annihilators;*
3. *R is a two-sided C_{11}-ring with ACC on right annihilators in which $Soc(R_R) = Soc(_RR)$ is essential as a left and a right ideal of R.*

Proof. (1) ⇒ (2) is clear.

(2) ⇒ (3) Since R is quasi-dual, $Soc(R_R) = Soc(_RR)$ is essential as a left and a right ideal of R by [25, Corollary 3.3].

(3) ⇒ (1) Since R has ACC on right annihilators and $Soc(R_R) = Soc(_RR)$ is essential as a left and a right ideal of R, we infer from [18, Lemma 4.3] that R is semiprimary. Thus, using [12, Lemma 4.2], we deduce that R is right Kasch. Hence,

by [12, Lemma 1.46], ${}_RR$ satisfies the C_2-condition. Now, we claim that R is right mininjective. To see this, let e be a local idempotent of R. Then $Soc(Re) \neq 0$. Since ${}_RR$ is a C_{11}-module satisfying the C_2-condition, it follows from [21, Proposition 2.3 (iii) and Theorem 4.3] that Re is a uniform module. Consequently, $Soc(Re)$ is simple. But $Soc(R_R) \subseteq Soc({}_RR)$. Then, R is right mininjective by [12, Proposition 3.5]. Therefore, by Theorem 2.2(2), R is quasi-Frobenius. $\square$

Corollary 3.2. *Then following conditions are equivalent for a ring R:*

1. *R is quasi-Frobenius;*
2. *R is right artinian two-sided C_{11} and $Soc(R_R) = Soc({}_RR)$.*

Corollary 3.3. *Then following conditions are equivalent for a ring R:*

1. *R is quasi-Frobenius;*
2. *R is two-sided C_{11} two-sided AGP-injective with ACC on right annihilators.*

Proof. (1) $\Rightarrow$ (2) is clear.

(2) $\Rightarrow$ (1) By [18, Theorem 3.4] and its proof, R is semiprimary and $Soc(R_R) = Soc({}_RR)$. Therefore, by Theorem 3.1(3), R is quasi-Frobenius. $\square$

The next example shows that the condition "$Soc(R_R) = Soc({}_RR)$" in the hypothesis of Corollary 3.2 is necessary.

Example 3.4 ([18, Remark 4.8(i)])**.** *Consider the ring $R = \begin{bmatrix} F & F \\ 0 & F \end{bmatrix}$, where F is a field. R is a two-sided artinian two-sided CS ring which is not quasi-Frobenius. However, $Soc(R_R) = \begin{bmatrix} 0 & F \\ 0 & F \end{bmatrix}$ and $Soc({}_RR) = \begin{bmatrix} F & F \\ 0 & 0 \end{bmatrix}$ and $Soc(R_R) \nleq Soc({}_RR)$ and $Soc({}_RR) \nleq Soc(R_R)$.*

4. Automorphism-invariant rings and their generalizations

Lemma 4.1. *If R is a left automorphism-invariant ring and containing no infinite orthogonal sets of idempotents, then R is semiperfect.*

Proof. Assume that R is a left automorphism-invariant ring and R contains no infinite orthogonal sets of idempotents. Let e be a primitive idempotent of R. Then, Re is an indecomposable autmorphism-invariant left R-module. It follows that $End(Re)$ is a local ring, and so e is a local idempotent of R. Thus, R is semiperfect. $\square$

Proposition 4.2. *If R is left automorphism-invariant and has ACC on right annihilators with $Soc({}_RR)$ an essential right ideal, then R is a quasi-Frobenius ring*

Proof. Assume that R is left automorphism-invariant and has ACC on right annihilators with $Soc({}_RR)$ an essential right ideal. Then, R is semiperfect by Lemma 4.1. Moreover, $J(R)$ is nilpotent by [9, Corollary 1.5]. It follows that R is semiprimary and so R is left self-injective. This shows that R is quasi-Frobenius. $\square$

Proposition 4.3. *Then following conditions are equivalent for a ring R:*

1. *R is quasi-Frobenius;*
2. *R is right automorphism-invariant right C_{11} with ACC on left annihilators.*

Proof. (1) $\Rightarrow$ (2) is clear.

(2) $\Rightarrow$ (1) Since R has ACC on left annihilators, it has enough idempotents. So, we can write $R_R = e_iR \oplus ... \oplus e_nR$ where each e_iR is a primitive orthogonal idempotent. Being autmomorphism-invariant, R_R is a C_3-modue by [14, page 26]. Thus, since R_R is a C_{11}-module, each e_iR is uniform by [21, Proposition 2.3 (iii)] and Theorem 4.3]. Therefore, according to the proof of ((5) $\Rightarrow$ (1) of [14, Theorem 2], R is right self-injective. Thus, using [12, Proposition 18.9], we deduce that R is quasi-Frobenius. $\square$

Corollary 4.4. *A left noetherian right automorphism-invariant C_{11}-ring is quasi-Frobenius.*

Recall from [13] that a module N is said to be pseudo M-c^*-injective if for any submodule A of M which is isomorphic to a closed submodule of M, every monomorphism from A to N can be extended to a homomorphism from M to N. A module M is called pseudo-c^*-injective if M is pseudo M-c^*-injective. A ring is called right pseudo-c^*-injective if R_R is pseudo-c^*-injective.

Proposition 4.5. *Then following conditions are equivalent for a ring R:*

(1) *R is quasi-Frobenius;*
(2) *R is left 2-injective with ACC on right annihilators and $Soc(_RR) \leq_e R_R$.*
(3) *R is left 2-injective right AGP-injective with ACC on right annihilators;*
(4) *R is left 2-injective right pseudo-c^*-injective with ACC on right annihilators.*

Proof. (1) $\Rightarrow$ (2), (3), (4) are clear.

(2) $\Rightarrow$ (1) Since R has ACC on right annihilators and $Soc(_RR) \leq_e R_R$, R is semiprimary by [18, Lemma 4.3]. Then by [12, Theorem 5.31], R is left Kasch. Consequently, R is right P-injective by [12, Lemma 5.21]. Therefore, by [12, Theorem 3.31], R is quasi-Frobenius.

(3) $\Rightarrow$ (2) Since R is right AGP-injective with ACC on right annihilators, R is semiprimary, by [25, Corollary 1.6]. Moreover, $J(R) = Z(R_R)$ by [25, Lemma 1.3], and so $Soc(R_R) \subseteq Soc(_RR)$. Hence, $Soc(_RR) \leq_e R_R$.

(4) $\Rightarrow$ (2) Since R is right pseudo-c^*-injective with ACC on right annihilators, it follows from [13, Corollary 3.6] that R is semirpimary. Hence, by [12, Theorem 5.31], $Soc(_RR) \leq_e R_R$. $\square$

A ring R is strongly right Johns if $M_n(R)$ is right Johns for all $n \geq 1$. By [12, Lemma 8.10], if $M_2(R)$ is right Johns, then so is R. We have the following result:

Corollary 4.6. *Then following conditions are equivalent for a ring R:*

(1) *R is quasi-Frobenius;*
(2) *R is strongly right Johns right pseudo-c^*-injective;*
(3) *R is strongly right Johns and $Soc(_RR) \leq_e R_R$;*
(4) *$M_2(R)$ is right Johns right pseudo-c^*-injective;*
(5) *$M_2(R)$ is right Johns and $Soc(_RR) \leq_e R_R$.*

Theorem 4.7. *Then following conditions are equivalent for a ring R:*

(1) *R is quasi-Frobenius;*
(2) *R is two-sided pseudo-c^*-injective, two-sided C_{11} and has ACC on right annihilators.*

Proof. (1) $\Rightarrow$ (2) is clear.

(2) $\Rightarrow$ (1) Since R is right pseudo-c^*-injective and has ACC on right annihilators, by [13, Corollary 3.6], R is semiprimary. Hence, we can write $R_R = e_iR \oplus ... \oplus e_nR$ where each e_iR is a primtive orthogonal idempotent. Being right pseudo-c^*-injective, R_R is a C_3-modue by [13, Theorem 3.1]. Thus, since R_R is a C_{11}-module, each e_iR is uniform by [21, Proposition 2.3 (iii) and Theorem 4.3]. Therefore, according to [13, Theorem 3.4], R is right continuous. Similary, since R is left C_{11}, we can easily show that R is left continuous. Now, being two-sided continuous with ACC on right annihilators, R is quasi-Frobenius by [18, Corollary 4.11]. □

5. More characterizations

In the next result, we a provide a necessary and sufficient condition for a left perfect right simple-injective ring to be quasi-Frobenius.

Theorem 5.1. *Then following conditions are equivalent for a ring R:*

(1) *R is quasi-Frobenius;*
(2) *R is left perfect right simple-injective and for every projective right R-module M, $Z_2(M)$ is injective;*
(3) *R is left perfect right simple-injective and for every injective right R-module M, $Z_2(M)$ is projective;*
(4) *R is left perfect right simple-injective and $Z(R_R)$ is a noetherian right R-module.*

Proof. (1) $\Rightarrow$ (2), (3), (4) are clear.

(2) $\Rightarrow$ (1) By [12, Theorem 2.21], $Soc(R_R) \subseteq Soc(_RR)$, from which it follows that $Soc(_RR) \leq_e R_R$. Using [12, Lemma 4.2], we deduce that R is left Kasch and $rl(T)$ is essential in a direct summand of R for all right ideals T of R. Also, R is right Kasch by [12, Theorem 3.12]. Therefore, according to [12, Proposition 6.14], $rl(T) = T$ for all right ideals T of R. Hence, $J(R) \leq Z_2(R_R)$ by [8, Lemma 2]. Let M be any projective R-module. Then, by [7, p. 48 Exercise 22], $M = Z_2(M) \oplus M'$ for some injective R-module. Therefore, by hypothesis, R is quasi-Frobenius.

(3) $\Rightarrow$ (1) Let M be an injective R-module. Thus, by the proof of (2) $\Rightarrow$ (1), $M = Z_2(M) \oplus M'$ for some projective R-module. By hypothesis, R is quasi-Frobenius.

(4) $\Rightarrow$ (1) As shown in the proof of (2) $\Rightarrow$ (1), R is left Kasch and $rl(T) = T$ for all right ideals T of R. Thus, by [12, Proposition 5.20], $Soc(_RR) \leq_e R_R$. It follows from [12, Corollary 5.53] that R is right finitely cogenerated. Using [12, Lemma 6.43], we deduce that $R/Z(R_R)$ is right noetherian. Note that $Z(R_R)$ is a noetherian right R-module. Hence, we infer from [12, Lemma 8.6] that right artinian. Finally, R is quasi-Frobenius by [12, Theorem 3.31]. □

Corollary 5.2. *Then following conditions are equivalent for a ring R:*

(1) *R is quasi-Frobenius;*
(2) *R is left perfect right self-injective and for every projective right R-module M, $Z_2(M)$ is injective;*
(3) *R is left perfect right self-injective and for every injective right R-module M, $Z_2(M)$ is projective.*

Recall that a ring R is said to be left pseudo-coherent if the left annihilator of every finite subset of R is finitely generated.

Theorem 5.3. *Then following conditions are equivalent for a ring R:*

(1) *R is quasi-Frobenius;*
(2) *R is two-sided minfull left (or right) pseudo-coherent and $J(R)$ is left (or right) T-nilpotent.*

Proof. (1) $\Rightarrow$ (2) is clear.

(2) $\Rightarrow$ (1) By [12, Corollary 5.53], $Soc(_RR)$ is a finitely generated right ideal. Note that R is left pseudo-coherent. Thus, $J(R)$ is finitely generated as a left ideal. Since $J(R)$ is left T-nilpotent, we infer from [12, Lemma 5.64] that R is right perfect. Therefore, according to [12, Lemma 6.50], R has ACC on left annihilators. On the other hand, $Soc(R_R) = Soc(_RR)$ is left finitely generated as a right R-module by [12, Corollary 5.53]. Hence, by [12, Lemma 3.30], R is right artinian and we conclude by [12, Theorem 3.31] that R is quasi-Frobenius. $\square$

Corollary 5.4. *Then following conditions are equivalent for a ring R:*

(1) *R is quasi-Frobenius;*
(2) *R is a dual left (or right) pseudo-coherent ring in which $J(R)$ is left (or right) T-nilpotent.*

Corollary 5.5. *Then following conditions are equivalent for a ring R:*

(1) *R is quasi-Frobenius;*
(2) *R is left perfect, two-sided mininjective and left (or right) pseudo-coherent.*

Theorem 5.6. *Let R be a right C_{11} right minifull ring such that $J^2(R) = r(A)$ for a finite subset A of R. Then $J(R)/J^2(R)$ is a finitely generated right R-module.*

Proof. Let $J^2(R) = r(a_1, ..., a_n)$. Define $\phi : R/J^2(R) \longrightarrow R^n_R$ via $\phi(a + J^2(R)) = r(a_1a, a_2a..., a_na)$ for $a \in R$. Then ϕ is a monomorphism. Hence, we may regard $J^2(R)/J(R)$ as a submodule of R^n_R. Also, we have $J(R)/J^2(R) = Soc(J(R)/J^2(R)) \subseteq Soc(R^n_R) = (Soc(R_R))^n$. On the other hand, $Soc(R_R)$ is finitely generated by Lemma 2.1. Therefore, as a direct summand of $(Soc(R_R))^n$, $J(R)/J^2(R)$ is a finitely generated right R-module. $\square$

Corollary 5.7. *Let R be a left perfect right C_{11} right mininjective ring. If $J^2(R) = r(A)$ for a finite subset A of R, then R is quasi-Frobenius.*

Proof. Since R is left perfect right mininjective, it is right minfull. Thus, $J(R)/J^2(R)$ is a finitely generated right R-module by Theorem 5.6. Now, being left perfect, R is right artinian by [4, Lemma 2.9]. Thus, using Corollary 2.5(5), we deduce that R is quasi-Frobenius. $\square$

The following theorem is motivated by Theorem 3.13 in [10]. First, we prove the following lemmas.

Lemma 5.8. *Let R be a left continuous ring right RMC. Then R is semiperfect.*

Proof. Assume that R is left continuous right RMC. Let $\overline{S}_1 = Soc(\overline{Q}_{\overline{Q}})$ where $\overline{Q} = R/J(R)$. By [8, Lemma 2], $\overline{Q}$ is a von Neumann regular left continuous ring. Consequently, $\overline{Q}/\overline{S}_1$ is von Neumann regular. In addition, since $\overline{Q}$ has right RMC, $\overline{Q}/\overline{S}_1$ has finite right uniform dimension by [5, Lemma 5.14]. It follows that $\overline{Q}/\overline{S}_1$ is semisimple. As $\overline{Q}$ is semiprime, then $\overline{S}_1 = Soc(_{\overline{Q}}\overline{Q})$. Thus, $\overline{Q}$ satisfies DCC on essential left ideals. Therefore, $\overline{Q}$ is an artinian ring by [5, Corollary 18.7(2)], and we conclude by [8, Lemma 2] that R is semiperfect. □

Lemma 5.9. *Let R be a left CS ring with right RMC such that every principal right ideal is right annihilator. Then $r(J(R))$ is a noetherian right R-module.*

Proof. Since every principal right ideal is right annihilator, R is a left C_2-ring by [12, Proposition 5.10]. Thus, by Lemma 5.8, R is semiperfect. Using [12, Theorem 5.52], we deduce that $r(J(R))$ is a noetherian right R-module, as required. □

Lemma 5.10. *Let R be a left CS ring with right RMC such that every principal right ideal is right annihilator. Then following conditions are equivalent:*

(1) *R is quasi-Frobenius;*
(2) *$Z(_RR) = Z(R_R)$.*

Proof. (1) $\Rightarrow$ (2) is clear.

(2) $\Rightarrow$ (1) By Lemma 5.9, $r(J(R))$ is a noetherian right R-module. By hypothesis, $Z(_RR) = Z(R_R)$. Thus, as $Z(_RR) = J$ by [8, Lemma 2], then it follows that $Soc(R_R)$ is right finitely generated. Therefore, according to [5, Lemma 5.14], R has finite right uniform dimension. Using [10, Proposition 2.4(e)], we deduce that $Z(R_R)$ is right artinian. Hence, by hypothesis, R has ACC on left annihilators. Clearly, R is right minannihilator by [12, Lemma 5.1] (i.e every minimal right ideal of R is an annihilator). Therefore, R is quasi-Frobenius by [12, Theorem 4.22((1) $\Leftrightarrow$ (2))]. □

Now, we are able to prove the following result which improve Theorem 3.13((1) $\Rightarrow$ (2) in [10] and Proposition 18.6 in [5].

Theorem 5.11. *The following conditions are equivalent for a ring R:*

(1) *R is quasi-Frobenius;*
(2) *R is a left P-injective left IN-ring with right RMC and $J(R)$ is nil-ideal;*
(3) *R is a left P-injective left IN-ring with right RMC and $Z(R_R) = Z(_RR)$.*

Proof. (1) $\Rightarrow$ (2) is clear.

(2) $\Rightarrow$ (3) Assume that R has the stated condition. By [10, Propositon 2.4(a)], $J(R)$ is nilpotent. It follows from [12, Proposition 5.10 and Theorem 6.32] and Lemma 5.8 that R is semiprimary. Since R is left P-injective, we infer from [12, Theorem 5.31] that $Z(R_R) = Z(_RR)$.

(3) $\Rightarrow$ (1) As R is a left IN-ring, it is left CS by [12, Theorem 6.32]. It is clear that every principal right ideal is right annihilator (for, R is left P-injective). But by hypothesis, $Z(_RR) = Z(R_R)$. Therefore, according to Lemma 5.10, R is quasi-Frobenius. □

Corollary 5.12. *The following conditions are equivalent for a ring R:*

(1) *R is quasi-Frobenius;*
(2) *R is a two-sided P-injective left IN-ring with right RMC.*

Proposition 5.13. *The following conditions are equivalent for a ring R:*

(1) *R is quasi-Frobenius;*
(2) *R is a right P-injective right IN-ring with right RMC.*

Proof. (1) $\Rightarrow$ (2) is clear.

(2) $\Rightarrow$ (1) By [12, Proposition 5.10 and Theorem 6.32], R is right continuous. Using [12, Proposition 18.14], we deduce that R is right artinian. Hence, R has ACC on right annihilators. Since R is left minannihilator, we infer from [12, Theorem 4.22((1) $\Leftrightarrow$ (2))] that R is quasi-Frobenius. $\square$

Proposition 5.14. *The following conditions are equivalent for a ring R:*

1. *R is quasi-Frobenius;*
2. *R is left Kasch, every closed right ideal is a right annihilator and $Z_2(R_R)$ is an injective artinian right R-module.*

Proof. (1) $\Rightarrow$ (2) is clear.

(2) $\Rightarrow$ (1) By [24, Theorem 10], R is semiperfect right continuous. Using [8, Lemma 2], we deduce that $J(R) \leq Z_2(R_R)$. Therefore, from the hypothesis, we can write $R = Z_2(R_R) \oplus K$, where K is a semisimple right ideal. It follows that R is quasi-Frobenius. $\square$

Let (P) be a property of rings. A ring R is called completely P if each factor ring of R has the property (P).

Proposition 5.15. *A left perfect right completely simple-injective ring is quasi-Frobenius.*

Proof. Let $\overline{R}$ be a factor ring of R. By the proof (2) $\Rightarrow$ (1) of Theorem 5.1, $\overline{R}$ is right continuous and $rl(T) = T$ for all right ideals T of R. It follows that $\overline{R}$ has finite right uniform dimension. Hence, every cyclic right R-module is finitely cogenerated. Thus, R is right artinian by [12, Lemma 1.52]. But R is two-sided mininjective. Therefore, R is quasi-Frobenius by [12, Theorem 3.31]. $\square$

Surjeet Singh and Yousef Al-Shaniafi (see [20, Theorem 1.10]) proved that: Let R be any commutative ring such that the injective envelope $E(R)$ of R is a projective R-module. Then $R = E(R)$, i.e., R is self-injective. From this, it is easy to see that for a commutative ring R satisfying ACC on annihilators such that the injective envelope $E(R)$ of R is a projective R-module then R is quasi-Frobenius. Now we will extend this result to the noncommutative case. A ring R is called *right duo* if every right ideal is an ideal.

For a subset X of a right R-module M over a ring R, we denote that $r_R(X)$ or $r(X)$ the right annihilator of X in R. Now let X and Y are two subset of a right R-module M, the subset $\{r \in R | Xr \subseteq Y\}$ of R is denoted by $[Y : X]$. Recall that if $Y \leq M_R$ then $[Y : X] \leq R_R$ and if $X \leq M_R$ and $Y \leq M_R$ then $[Y : X]$ is an ideal of R.

Let R be a right duo ring and P be a maximal ideal of R. Then it is easy to prove that $R \setminus P$ is multiplicatively closed and satisfies condition (S1): $\forall s \in R \setminus P$ and $r \in R$, there exist $t \in R \setminus P$ and $u \in R$ such that $su = rt$. Moreover, if R satisfies ACC on right annihilators then by [17, Proposition 1.5], $R \setminus P$ is a right denominator set. In this case, the ring $R(R \setminus P)^{-1}$ is called the *right localization with respect to* P and we write R_P and M_P instead of $R(R \setminus P)^{-1}$ and $M(R \setminus P)^{-1} = M \otimes_R R_P$, respectively. A ring R is called *right localizable* if for each maximal right ideal P of R, the right localization R_P exists. A ring R is said to be left quasi-duo if each of its maximal left ideals is an ideal of R. A ring R is called right QF-3^+ if the injective envelope $E(R_R)$ of R_R is a projective right R-module.

Theorem 5.16. *Let R be a right duo, right QF-3^+, left quasi-duo ring satisfying ACC on right annihilators. Then R is quasi-Frobenius.*

Proof. Now let P be a maximal ideal of R and $\theta : E \to E_P$ be the canonical map. Then the right localization R_P exists. Since E is projective, we have $E \oplus A = R^{(X)}$ with some A_R and index set X. We know that $E_P = E \otimes_R R_P$, so

$$\begin{aligned}(E \oplus A) \otimes_R R_P &= (E \otimes_R R_P) \oplus (A \otimes_R R_P) \\ &= R^{(X)} \otimes_R R_P \cong R_P^{(X)}\end{aligned}$$

Hence E_P is a projective right R_P-module.

Let $F = \{x \in E | [EP : x] \nsubseteq P\}$. With assumption $\theta(1) \in E_P P$ and by [19, Lemma 3.17], $[EP : 1] \nsubseteq P$. So $1 \in F$. Similarly, by [19, Lemma 3.17], $\theta(x) \in E_P P$ if and only if $[EP : x] \nsubseteq P$. So $F = \{x \in E | \theta(x) \in E_P P\}$. Because θ is an R-homomorphism, we can prove easily that F is a submodule of E.

Now we will prove that F is quasi-injective. Now since $E(F)$ is a direct summand of E, we can assume that we take any homomorphism $\psi : E \longrightarrow E$. There exists an R_P-homomorphism $\sigma : E_P \longrightarrow E$ such that $\sigma\theta = \psi$, i.e., the following diagram is commutative:

$$\begin{tikzcd} E \arrow[r, "\theta"] \arrow[dr, "\psi"'] & E_P \arrow[d, "\sigma"] \\ & E \end{tikzcd}$$

Now, let $t \in F$ then $t \in E$ and there exists $r \notin P$ such that $tr \in EP$. Moreover, $\theta(t) \in E_P P$. Hence there exists $p \in P, e_t \in E_p$ such that $\theta(t) = e_t p$. So $\psi(t) = (\sigma\theta)(tr) = \sigma(\theta(t))r = (\sigma\theta)(e_t p)r = (\sigma\theta)(e_t)pr \in EP$. It follows that $\psi(t) \in L$.

Since F is invariant under any homomorphism of E, F is quasi-injective. Now since $1 \in F$, there exist $r \in EP$ such that $r \notin P$. Let $e \in E$ then since $r \in (EP) \cap R$, $er \in E[(EP) \cap R] \leq EP$. So $e \in F$. Hence $E = F$. Hence $E_P \neq E_P P$. So there exists an $e \in E$ such that $\theta(e) \notin E_P P$. Since $E = L$, $e \in L$, so $[EP : e] \nsubseteq P$. Then there exists $v \notin P$ such that $ev \in EP$. Hence $\theta(e) \in EP$. Contradiction. Hence $\theta(1) \notin E_P P$. Since R_P is a local ring and E_P is a non-zero projective R_P-module, so it is free and then

$$E_P = \bigoplus_{i \in I} A_i, \quad A_i \cong R_P.$$

Now we prove that E/R is a flat right R-module. By [17, Exe. 39, p. 48] we need to prove that for every maximal left ideal P of R, $EP \neq E$. Note that P is an ideal and since $\theta(1) \notin E_P P$, $R \cap EP \leq P$. Assume that $EP = E$ then $x \in R \Rightarrow x \in E \Rightarrow x \in EP \Rightarrow x \in P$. So $R = P$. Contradiction. Since E is projective and by [12, Lemma 7.30], E is also finitely generated, so for some $n \in \mathbb{N}$, we obtain that $R^n \to E/R \to 0$ is exact and then by [17, Cor. 11.4, p.38], E/R is projective. Then $E = R$. And R is right self-injective. Then R is quasi-Frobenius.

Corollary 5.17. *(*[20, Theorem 1.10]*) Let R be any commutative, QF-3^+ ring satisfying ACC on annihilators. Then R is quasi-Frobenius.*

Acknowledgments. Le Van Thuyet and Truong Cong Quynh acknowledge the support/partial support of the Core Research Program of Hue University, Grant No. NCM.DHH.2020.15. Parts of this paper were written during a stay of Le Van Thuyet and Truong Cong Quynh in the Vietnam Institute For Advanced Study in Mathematics (VIASM) and they would like to thank the members of VIASM for their hospitality, as well as to gratefully acknowledge the received support.

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